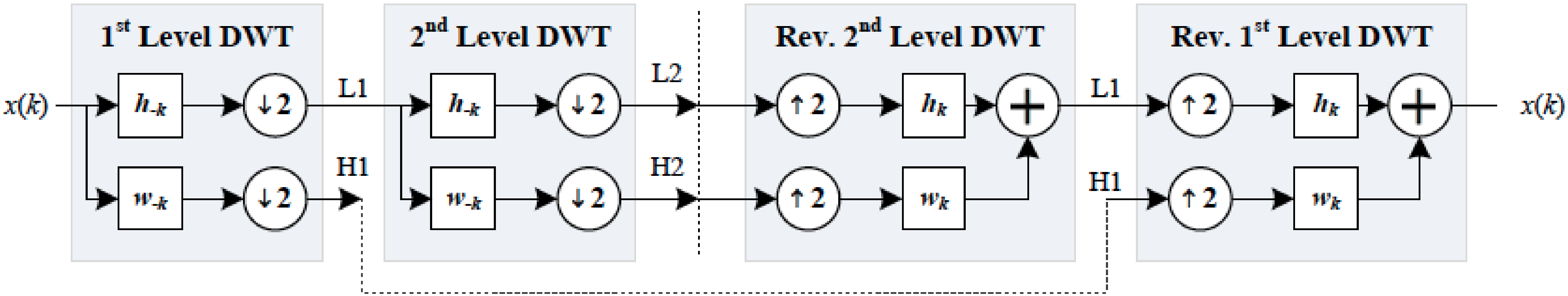
ECE462 – Lecture 15

DWT Review



- Forward DWT performed by iteratively filtering and downsampling (discarding every other sample)
  - h_{-k} (low-pass) and w_{-k} (high-pass) filters are time-reversed (flipped) version of filters used for the reverse transform (h_k and w_k)
  - L1 output is *N*/2 in length, representing lower-resolution version of input – called low-pass subband
  - H1 output is *N*/2 in length, representing details missing from L1 (e.g., sharp edges) – called high-pass subband
  - Each level uses previous low-pass output as new input

DWT Review (cont'd)

- Reverse DWT performed by upsampling (inserting zeroes after each input sample), filtering, and adding
- Note: filtering (linear convolution) involves zero-padding the signal and results in expansion of output signal length ($N+M-1$, where M is length of filter) – methods to avoid this are beyond scope of this course
  - For the Haar filter ($M=2$), this is easily avoided by not zero-padding the beginning of the signal for forward transform, and not zero-padding the end of the signal during reverse transform (see example)
- Maximum number of levels = $\log_2 N$
  - For a image of size, e.g., 512x512, we may typically perform between 4 and 6 DWT levels in compression systems
- Each DWT level represents a "band" (range) of frequencies (rather than a single frequency, like DCT coefficients) – trade-off is that each subband retains spatial information about the input
  - Each subsequent level narrows the band of frequencies (i.e., providing more specific frequency information), resulting in lower spatial resolution

1-D DWT Example

- 1-D, 2-Level Haar Example: $x(k)$ = [1,2,3,4,5,6,7,8]

$$h_k = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \quad w_k = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$

1st Level

$$x(k) \xrightarrow{h_{-k}} \downarrow 2 \Rightarrow \mathrm{L1} = \left[\frac{1+2}{\sqrt{2}}, \frac{3+4}{\sqrt{2}}, \frac{5+6}{\sqrt{2}}, \frac{7+8}{\sqrt{2}}\right]$$

$$x(k) \xrightarrow{w_{-k}} \downarrow 2 \Rightarrow \mathrm{H1} = \left[\frac{1-2}{\sqrt{2}}, \frac{3-4}{\sqrt{2}}, \frac{5-6}{\sqrt{2}}, \frac{7-8}{\sqrt{2}}\right]$$

$$\therefore x_1 = \left[\frac{3}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{11}{\sqrt{2}}, \frac{15}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right]$$

2nd Level

$$\mathrm{L1} \xrightarrow{h_{-k}} \downarrow 2 \Rightarrow \mathrm{L2} = \left[\frac{3+7}{2}, \frac{11+15}{2}\right]$$

$$\mathrm{L1} \xrightarrow{w_{-k}} \downarrow 2 \Rightarrow \mathrm{H2} = \left[\frac{3-7}{2}, \frac{11-15}{2}\right]$$

$$\therefore x_2 = \left[5, 13, -2, -2, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right]$$

1-D DWT Example (cont'd)

- Reverse 1-D, 2-Level Haar Example:

$$x_2(k) = \left[5, 13, -2, -2, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right] \qquad h_k = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \quad w_k = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$

2nd Level

$$\mathrm{L2} \Rightarrow \uparrow 2 = [5,0,13,0] \xrightarrow{h_k} \left[\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{13}{\sqrt{2}}, \frac{13}{\sqrt{2}}\right]$$

$$\mathrm{H2} \Rightarrow \uparrow 2 = [-2,0,-2,0] \xrightarrow{w_k} \left[\frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}}, \frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right]$$

Add

$$\therefore \mathrm{L1} = \left[\frac{3}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{11}{\sqrt{2}}, \frac{15}{\sqrt{2}}\right] \Rightarrow x_1 = \left[\frac{3}{\sqrt{2}}, \frac{7}{\sqrt{2}}, \frac{11}{\sqrt{2}}, \frac{15}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right]$$

1st Level

$$\mathrm{L1} \Rightarrow \uparrow 2 = \left[\frac{3}{\sqrt{2}}, 0, \frac{7}{\sqrt{2}}, 0, \frac{11}{\sqrt{2}}, 0, \frac{15}{\sqrt{2}}, 0\right] \xrightarrow{h_k} \left[\frac{3}{2}, \frac{3}{2}, \frac{7}{2}, \frac{7}{2}, \frac{11}{2}, \frac{11}{2}, \frac{15}{2}, \frac{15}{2}\right]$$

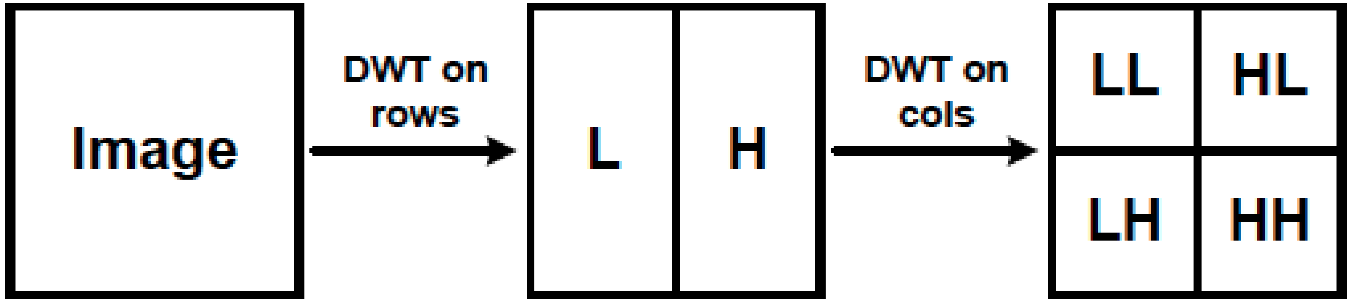
$$\mathrm{H1} \Rightarrow \uparrow 2 = \left[\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}, 0\right] \xrightarrow{w_k} \left[\frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}\right]$$

Add

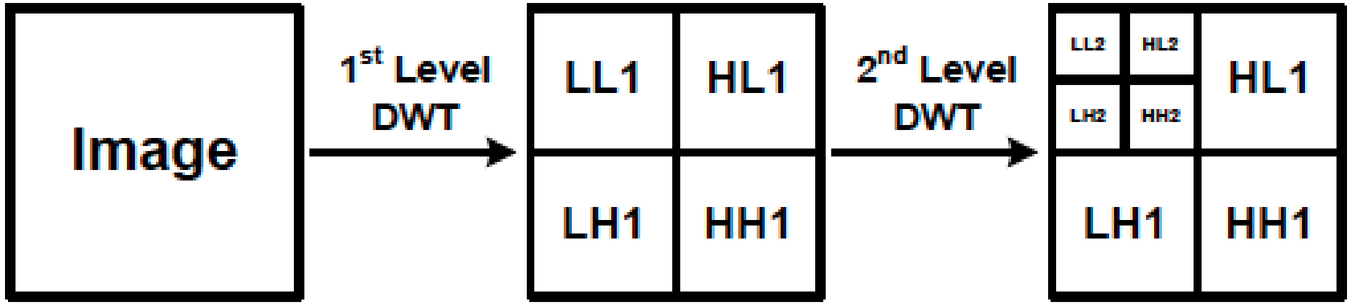
$$\therefore x(k) = [1,2,3,4,5,6,7,8]$$

2-D DWT Review

- Forward 2-D DWT performed by applying 1-D transform separably
  - First perform transform on rows (or columns) then on columns (or rows)
  - Subbands labelled LL, LH, HL, HH, depending on whether low-pass or high-pass filter was applied in horizontal or vertical direction



  - Each level is performed by using the previous level LL subband as new input



2-D DWT Example

$$x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad h_k = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right], \quad w_k = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$

- Perform transform on rows, then columns

$$L1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \end{bmatrix} \Rightarrow \begin{matrix} LL1 = \begin{bmatrix} 1 & 0 \\ 3/2 & 1 \end{bmatrix} \\ LH1 = \begin{bmatrix} 0 & 0 \\ -1/2 & -1 \end{bmatrix} \end{matrix} \qquad H1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} HL1 = \begin{bmatrix} 1 & 0 \\ 1/2 & 0 \end{bmatrix} \\ HH1 = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix} \end{matrix}$$

$$\therefore x_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2 & -1 & 1/2 & 0 \end{bmatrix}$$

1st Level

$$L2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{5}{2\sqrt{2}} \end{bmatrix} \Rightarrow \begin{matrix} LL2 = [7/4] \\ LH2 = [-3/4] \end{matrix} \qquad H2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \Rightarrow \begin{matrix} HL2 = [3/4] \\ HH2 = [1/4] \end{matrix}$$

$$\therefore x_2 = \begin{bmatrix} 7/4 & 3/4 & 1 & 0 \\ -3/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2 & -1 & 1/2 & 0 \end{bmatrix}$$

2nd Level

2-D DWT Example (cont'd)

- Perform reverse transform

$$x_2 = \begin{bmatrix} 7/4 & 3/4 & 1 & 0 \\ -3/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2 & -1 & 1/2 & 0 \end{bmatrix} \qquad h_k = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \qquad w_k = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$

$$LL2 \Rightarrow \uparrow 2 = \begin{bmatrix} 7/4 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{h_k(rows)} \begin{bmatrix} \frac{7}{4\sqrt{2}} & \frac{7}{4\sqrt{2}} \\ 0 & 0 \end{bmatrix} \xrightarrow{h_k(cols)} \begin{bmatrix} 7/8 & 7/8 \\ 7/8 & 7/8 \end{bmatrix}$$

$$LH2 \Rightarrow \uparrow 2 = \begin{bmatrix} -3/4 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{h_k(rows)} \begin{bmatrix} \frac{-3}{4\sqrt{2}} & \frac{-3}{4\sqrt{2}} \\ 0 & 0 \end{bmatrix} \xrightarrow{w_k(cols)} \begin{bmatrix} -3/8 & -3/8 \\ 3/8 & 3/8 \end{bmatrix}$$

$$HL2 \Rightarrow \uparrow 2 = \begin{bmatrix} 3/4 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{w_k(rows)} \begin{bmatrix} \frac{3}{4\sqrt{2}} & \frac{-3}{4\sqrt{2}} \\ 0 & 0 \end{bmatrix} \xrightarrow{h_k(cols)} \begin{bmatrix} 3/8 & -3/8 \\ 3/8 & -3/8 \end{bmatrix}$$

$$HH2 \Rightarrow \uparrow 2 = \begin{bmatrix} 1/4 & 0 \\ 0 & 0 \end{bmatrix} \xrightarrow{w_k(rows)} \begin{bmatrix} \frac{1}{4\sqrt{2}} & \frac{-1}{4\sqrt{2}} \\ 0 & 0 \end{bmatrix} \xrightarrow{w_k(cols)} \begin{bmatrix} 1/8 & -1/8 \\ -1/8 & 1/8 \end{bmatrix}$$

Add

$$\therefore LL1 = \begin{bmatrix} 1 & 0 \\ 3/2 & 1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2 & -1 & 1/2 & 0 \end{bmatrix}$$

2-D DWT Example (cont'd)

- Perform reverse transform (cont'd)

$$x_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3/2 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2 & -1 & 1/2 & 0 \end{bmatrix}$$

$$\text{LL1} \Rightarrow \uparrow 2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{h_k(rows)} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3/2\sqrt{2} & 3/2\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{h_k(cols)} \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 3/4 & 1/2 & 1/2 \\ 3/4 & 3/4 & 1/2 & 1/2 \end{bmatrix}$$

$$\text{LH1} \Rightarrow \uparrow 2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{h_k(rows)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/2\sqrt{2} & -1/2\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_k(cols)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/4 & -1/4 & -1/2 & -1/2 \\ 1/4 & 1/4 & 1/2 & 1/2 \end{bmatrix}$$

$$\text{HL1} \Rightarrow \uparrow 2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_k(rows)} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2\sqrt{2} & -1/2\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{h_k(cols)} \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 1/4 & -1/4 & 0 & 0 \\ 1/4 & -1/4 & 0 & 0 \end{bmatrix}$$

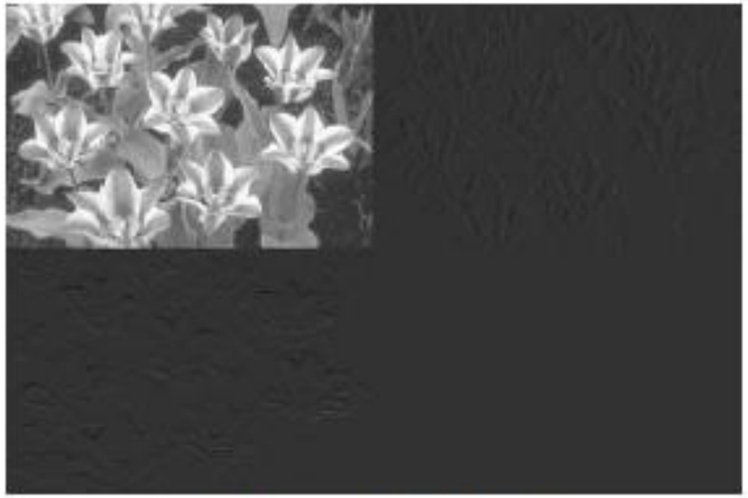
$$\text{HH1} \Rightarrow \uparrow 2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_k(rows)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2\sqrt{2} & -1/2\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{w_k(cols)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & -1/4 & 0 & 0 \\ -1/4 & 1/4 & 0 & 0 \end{bmatrix}$$

Add

$$x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

2-D DWT Image Example

Image



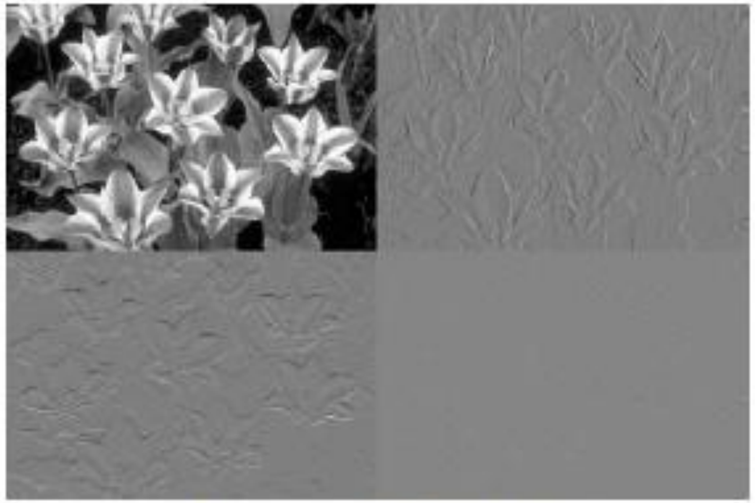
1st Level DWT

2nd Level DWT

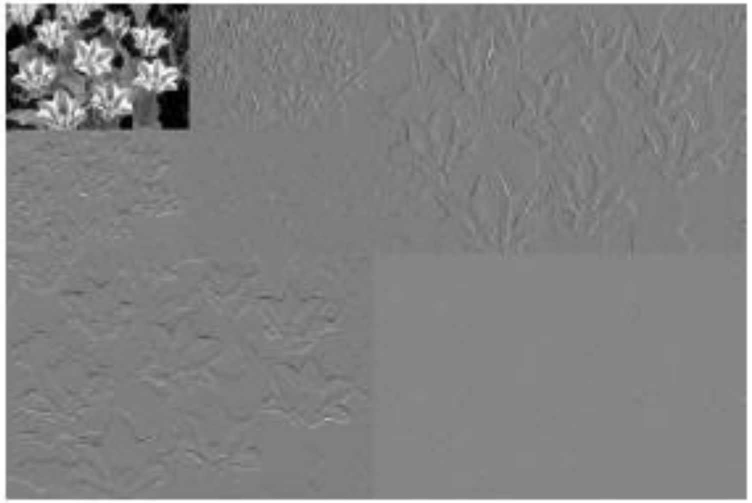
3rd Level DWT

2-D DWT Image Example

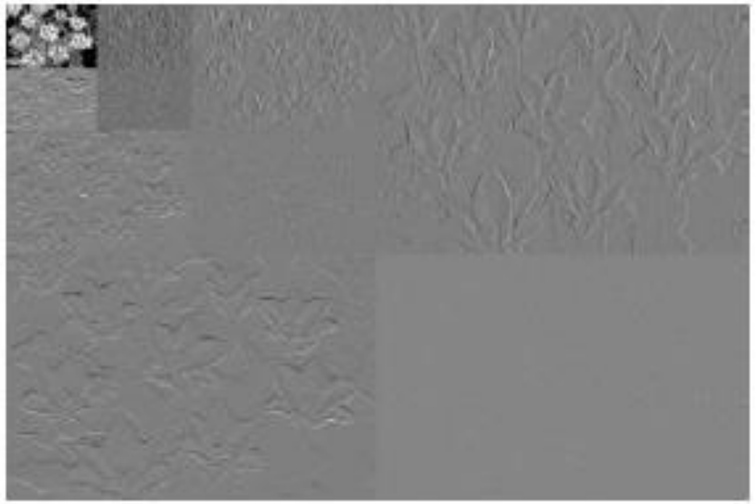
Increase contrast to see detail in high-frequency subbands



1st Level DWT



2nd Level DWT



3rd Level DWT