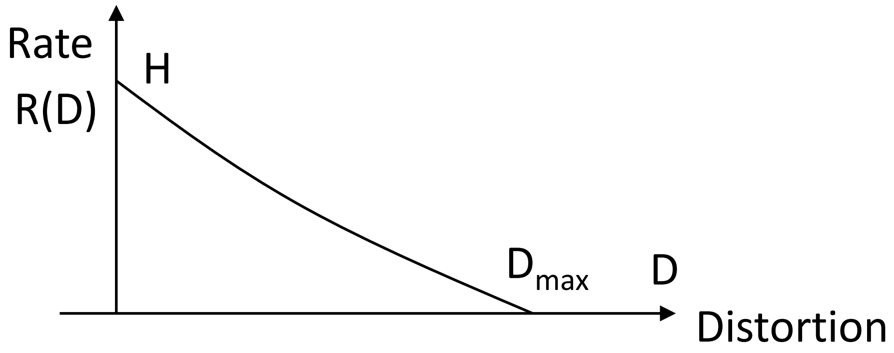
ECE462 – Lecture 17

- **Bit allocation and Quantization in wavelet coding**

  - Given M sub-images and a fixed bit rate R how do we assign bits to the sub-images?
  - Define a cost function D in terms of bpp/sub-image and minimize it or maximize it w.r.t the number of bits R. D may represent distortion, energy or entropy.
  - Rate distortion theory provides a bound or limit of the achievable rate as a function of distortion.
  - The objective is to allocate bits so that we get as close to the curve as possible



- **Example:**

  Consider a 4-level wavelet decomposition of an image → M=13 sub-bands

  Image size: 256x256 pixels

  Smaller sub-band: 16x16 pixels

  Larger sub-band: 128x128 pixels

  Let:

  - N: total number of bits in original image (256x256x8 = 524,288)
  - M: number of sub-bands (M=13)
  - $a_k = \frac{N_k}{N}$ the relative sub-band size
  - b_k , k=1,…,M bit lengths in sub-band k
  - 6_k^2 : variance of values in sub-band k
  - w_k : perceptual coding weighting factor in sub-band k

Let b = (b_1, b_2,……, b)

Then we can define the quantization error power as:

$$D(b) = \sum_{k=1}^{M} a_k w_k \; 2^{-2b_k} \, 6_k^{2}$$

Notice that the error is white random process with it's power ~ $\frac{6_k^{2}}{{}_{2}2b_k}$

Also let the total bit rate is:

$$R(b) = \sum_{k=1}^{M} a_k b_k$$

Then the bit allocation problem is:

Minimize D(b) subject to R(b) = Rc

Where Rc is a desired fixed rate

- **<u>Optimization method using Lagrange multiplier</u>**

For the solution we can use a Lagrange multiplier λ and minimize $\{D(b) + \lambda R(b)\}$ w.r.t $\{b_k\}$

$$\sum_{k=1}^{M} a_k(\ w_k \ 2^{-2b_k} \sigma_k^2 + \lambda b_k)$$

Min $\sum_{k=1}^{M} a_k(w_k \ 2^{-2b_k} \sigma_k^2 + \lambda b_k) \rightarrow \min (w_k 2^{-2b_k} \sigma_k^2 + \lambda b_k)$, k=1,…,M

Differentiate w.r.t b_k and set to 0

$$w_k (-2) \log_e 2 \, e^{-2b_k \log_e 2} \sigma_k^2 + \lambda = 0$$

$$\Rightarrow e^{-2b_k \log_e 2} \sigma_k^2 = \frac{\lambda}{w_k \log_e 2 \, \sigma_k^2} \Rightarrow 2 b_k \log_e 2 = \log_e \frac{2 w_k \log_e 2 \, \sigma_k^2}{\lambda}$$

$$\Longrightarrow \boxed{b_k = \frac{1}{2}\ \log_2 \frac{2\ \log_e 2\ w_k 6_k{}^2}{\lambda}}$$

Then, $\sum_{k=1}^{m} a_k b_k = \frac{1}{2} \sum_{k=1}^{m} a_k\ \log_2 \frac{(2\ \log_e 2)\ w_k 6_k{}^2}{\lambda}\ = R_c$ (please replace m with M)

$$\Longrightarrow \boxed{\lambda = 2^{\{\sum_{k=1}^{m} a_k \log_2 ((2\ \log_e 2)\ w_k 6_k{}^2\ ) - 2Rc\}}}$$

Finally: $b_k = \frac{1}{2}\ \log_2 2\ \log_e 2\ \ w_k 6_k{}^2 - \frac{1}{2} \sum_{k=1}^{m} a_k\ \log_2 2\ \log_e 2\ \ w_k 6_k{}^2\ + Rc$

$$\Longrightarrow b_k = R_c + \frac{1}{2}\ \log_2\ w_k 6_k{}^2 - \frac{1}{2} \sum_{k=1}^{m} a_k\ \log_2 w_k 6_k{}^2$$

$$\Longrightarrow \boxed{b_k = R_c + \frac{1}{2}\ \log_2 \frac{w_k 6_k{}^2}{\prod_{k=1}^{M} (w_k 6_k{}^2)^{a_k}}}$$

- **The find algorithm proceeds as follows:**

1. Estimate 6_k^2 for each sub-band k. Given R_c , wk obtain λ
2. Estimate b_k , k=1,2,…,M if $b_k < 0$ then place $b_k = 0$

3. Reduce number of sub-bands to M_{new}
   (corresponding to non-zero b_k)

4. Adjust if desired Rc to Rc,new. Repeat until all $b_k \geq 0$

Now from 6_k and b_k, we can compare the step size Δ_k for each quantizer in sub-band k.

If $b_k = 0 : \Delta_k = 2\, X_{k,max}$

Where $X_{k,max}$ is the largest non-zero value in sub-band k

If $b_k > 0 : \Delta_k = \max\left(\frac{2 6_k}{2^{b_k}}, \frac{X_{k,max}}{2^{b_{k,max}}}\right)$

- **Numerical application:**

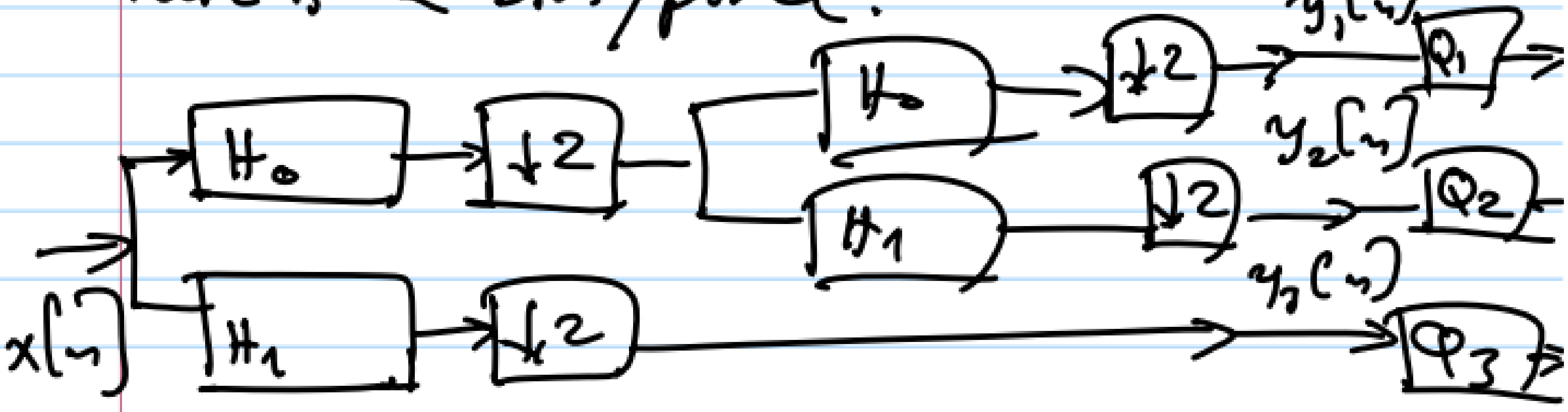
An image is modelled as a stationary, one-dimensional, zero mean Gaussian signal x[n] with an auto correlation function

$$R_x [k] = E\{x[n]x[n+k] = 2x0.85^{|k|}$$

We want to process the image using a two level wavelet transform and quantize the WT coefficients so that the resulting rate is 2 bits/pixel.

- **Numerical application:**

rate is 2 bits/pixel.



where, $H_0(z) = \frac{1}{\sqrt{2}}\left(1 + z^{-1}\right)$

$$H_1(z) = \frac{1}{\sqrt{2}}\left(1 - z^{-1}\right)$$

A) Calculate the variances of the signals $y_1[n], y_2[n], y_3[n]$

$$Var[y_3[n]] = E\left[\left(\frac{x[n+1]-x[n]}{\sqrt{2}}\right)^2\right] = \frac{1}{2}[2E\{x^2[n]\} - 2E\{x[n+1]x[n]\}]$$

$$= R(0) - R(1) = 2(1-0.85) = 0.3$$

$$Var[y_1[n]] = E\left[\left(\frac{x[n+3]+x[n+2]}{\sqrt{2}} + \frac{x[n+1]+x[n]}{\sqrt{2}}\right)^2\right]$$

$$= \frac{1}{4}[4E\{x[n]^2\}] + 6E[\{x[n+1]x[n]\}] + 4E[\{x[n+2]x[n]\}] + 2E[\{x[n+3]x[n]\}] = \cdots \ldots \ldots .. = 6.61$$

$$Var\big[y_2[n]\big] = E\left[\frac{1}{\sqrt{2}}\left(\frac{x[n+3]+x[n+2]}{\sqrt{2}} - \frac{x[n+1]+x[n]}{\sqrt{2}}\right)^2\right]$$

= …. = 0.731

B) Now for $y_1[n], k = 1, a_1 = \frac{1}{4}, \sigma_1^2 = 6.61$

$$y_2[n], k = 2, a_2 = \frac{1}{4}, \sigma_2^2 = 0.731$$

$$y_3[n], k = 3, a_3 = \frac{1}{2}, \sigma_3^2 = 0.3$$

$$R_c = 2\ bits/sample$$

- Thus according to the formula for

$$b_k = R_c + \frac{1}{2}\log_2 \frac{\sigma_k^2}{\pi_{k=1}^{3}\sigma_k^{2a_k}}$$

$$b_k = 2 + \log_2 \frac{\sigma_k^2}{\sigma_1^{\frac{1}{2}}\sigma_2^{\frac{1}{2}}\sigma_3} = 8 + \log_2 \frac{\sigma_k^2}{0.8120}$$

$$b_1 = 2 + \frac{1}{2}\log_2 \frac{6.61}{0.8120} \cong 3.5 \rightarrow 4\ bit$$

$$b_2 = 2 + \frac{1}{2}\log_2 \frac{0.731}{0.8120} \cong 1.92 \rightarrow 2\ bit$$

$$b_3 = 2 + \frac{1}{2}\log_2 \frac{0.3}{0.8120} \cong 1.28 \rightarrow 1\ bit$$

- Thus:

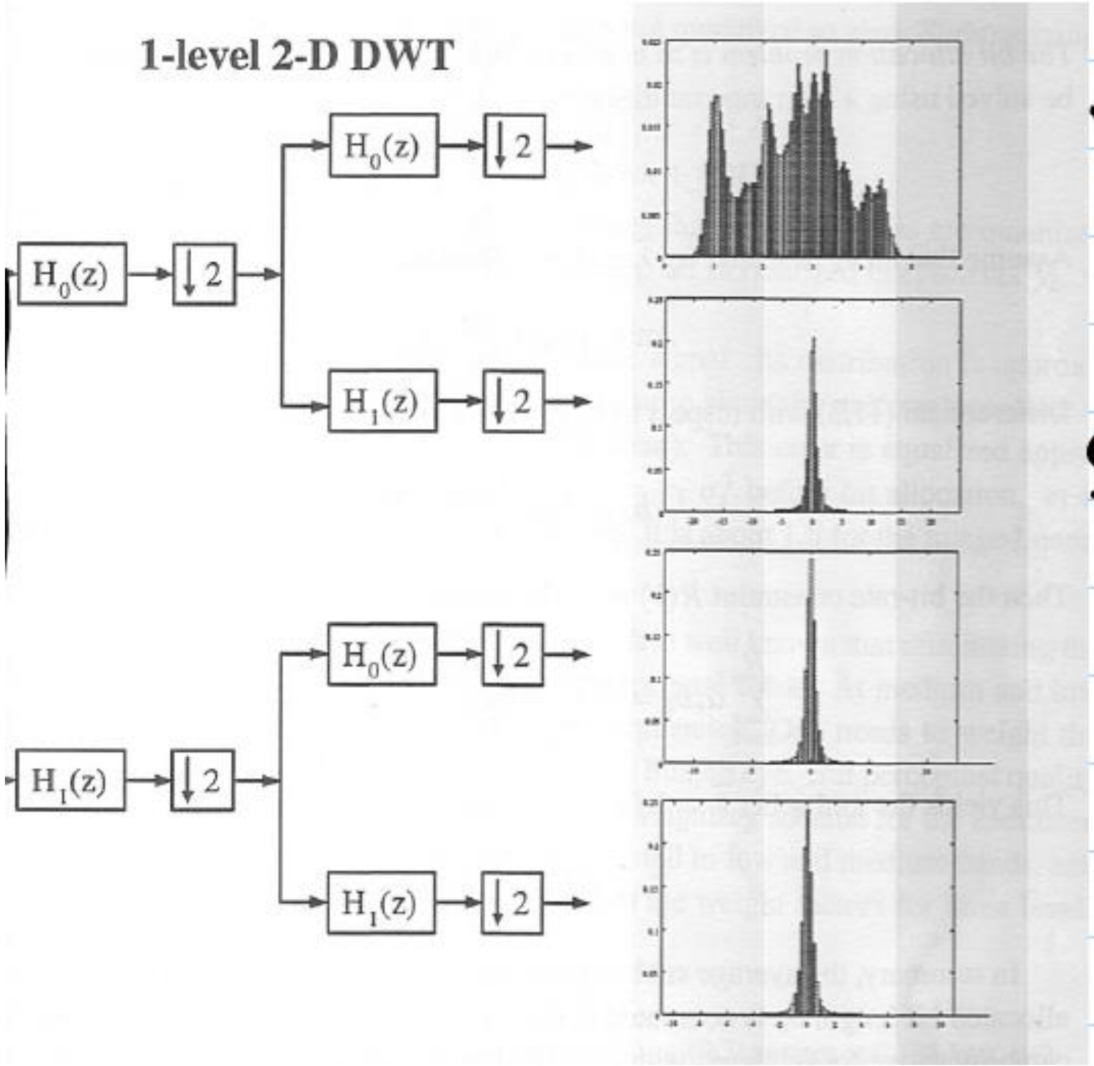
Q1 gets 4 bits/sample

Q2 gets 2 bits/sample

Q3 gets 1 bit/sample

Total: $R_c = \frac{1}{4}.4 + \frac{1}{4}.2 + \frac{1}{2}.1 = 1 + \frac{1}{2} + \frac{1}{2} = 2\ bits/sample$

Numerical Application



Subband 1 ~ uniform distribution

$\sigma^2 \sim 4/3$

$X_{MAX} \sim 5$

Subbands 2–4 ~ Gaussian distribution

$\sigma^2 \sim 0.3$

$X_{MAX} \sim 5$

So in this case,

$$M = 4, a_1 = \frac{N_1}{N} = \frac{1}{4}, a_2 = a_3 = a_4 = \frac{1}{4}, \sigma_1^2 = \frac{4}{3}, \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = 0.3$$

Let us choose $W_1 = W_2 = W_3 = W_4 = 1$

Then: λ= $2^{\left\{\frac{1}{4}\left[\log_2 2\left(\frac{0.6931.4}{3}\right)+3\log_2 2.0.6931.0.3-2\,R_c\right]\right\}} = 2^{\{-0.7278-2R_c\}}$

Let $R_c = 0.75 \rightarrow \lambda = 0.2135$

$$so\ b_1 = \frac{1}{2}\log_2 \frac{1\ldots.62.\frac{4}{3}}{0.2135} = 1.557$$

$$b_2 = \frac{1}{2}\log_2 \frac{1.3862.0.3}{0.2135} = 0.48$$

$$b_3 = b_4 = b_2 = 0.48$$

Then, $\delta_1 = MAX\left(\frac{2^{\sqrt{\frac{4}{3}}}}{2^{1.557}}, \frac{5}{2^{8}-1}\right) = MAX(0.7849, 0.0196) = 0.7849$

$$\delta_2 = \delta_3 = \delta_4 = MAX\left(\frac{2^{\sqrt{0.3}}}{2^{0.48}}, \frac{5}{2^8}\right) = MAX(0.7854,0.0110) = 0.7854$$

Now let $R_c = 0.25 \rightarrow ב = 0.4270, b_1 = 1.057,$

$$b_2 = b_3 = b_4 = -0.191 = 0$$

So then, $R_c = R(b) = \frac{1}{4}. b_1 = 0.25 \Rightarrow b_1 = 1 \; and \; \delta_1 = 1.1547$