MULTIRATE SYSTEMS

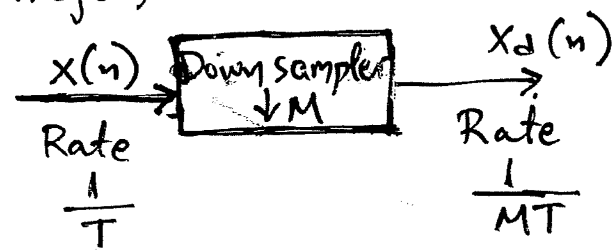
"The sampling rate is not constant throughout the system"

Decimation { filtering + down-sampling operation

Interpolation: up-sampling operation + filtering

- M-fold decimation (M integer)

Let $\boxed{x_d(n) = x(Mn)}$



Let $x(n) = x_a(nT)$ then $x_d(n) = x(Mn) = x_a(nMT)$

From sampling theorem

$$X(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi k}{T}\right) \qquad (*)$$

and

$$X_d(\omega) = \frac{1}{MT}\sum_{l=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi l}{MT}\right) = \frac{1}{MT}\sum_{l=Mr+i} X_a\left(\frac{\omega - 2\pi(Mr+i)}{MT}\right)$$

where $r = 0, \pm 1, \pm 2, \dots,\ \ i = 0, 1, \dots, M-1$.

$$\Rightarrow X_d(\omega) = \frac{1}{MT}\sum_{r=-\infty}^{\infty}\sum_{i=0}^{M-1} X_a\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi r}{T}\right) =$$

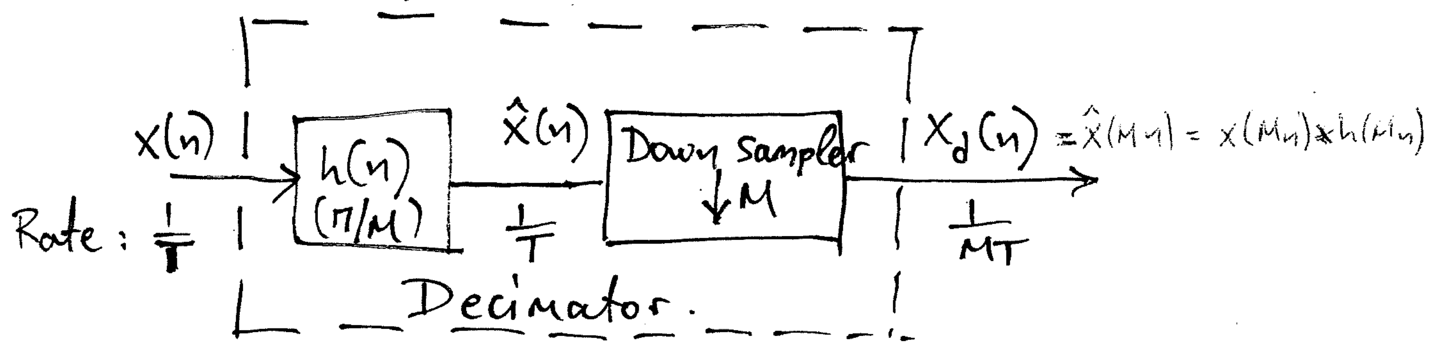
$$= \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi i}{MT} - \frac{2\pi r}{T}\right)\right] \Rightarrow \text{from } (*)$$

$$\boxed{X_d(\omega)=\frac{1}{M}\sum_{i=0}^{M-1} X\left(\frac{\omega-2\pi i}{M}\right)} \iff \boxed{x_d(n)=x(Mn)}$$

Expansion + aliasing $\longleftrightarrow$ Compression

* To avoid aliasing $X(\omega)$ must be bandlimited in the range $-\frac{\pi}{M}\leq\omega\leq\frac{\pi}{M}$ (lowpass signal)

Otherwise a filter should be placed before the down-sampler.
(decimation filter)



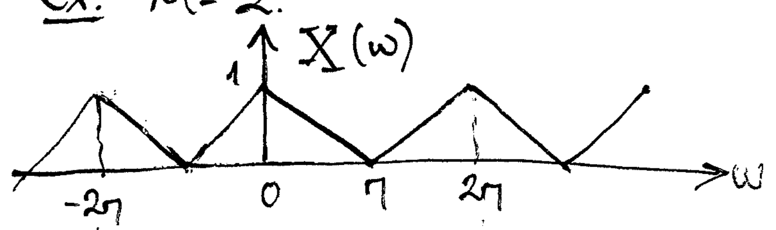
Linear but time variant system.

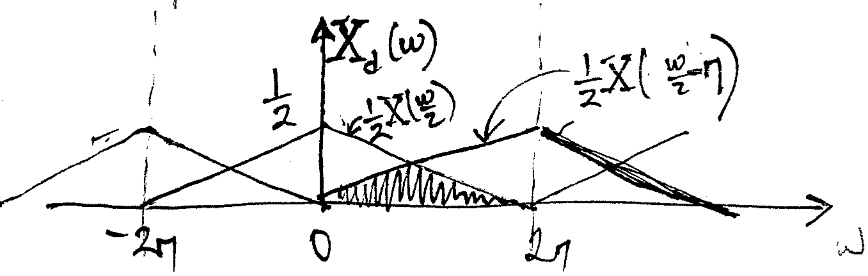
In this case: $X_d(\omega)=\frac{1}{M}\sum_{i=0}^{M-1}\hat{X}\left(\frac{\omega-2\pi i}{M}\right)=\frac{1}{M}\sum_{i=0}^{M-1}H\left(\frac{\omega-2\pi i}{M}\right)X\left(\frac{\omega-2\pi i}{M}\right)$

assuming no aliasing $=\frac{1}{M}H\left(\frac{\omega}{M}\right)X\left(\frac{\omega}{M}\right)$, $-\pi\leq\omega\leq\pi$

Ex: $M=2$.

$$X_d(\omega)=\frac{1}{2}\left[X\left(\frac{\omega}{2}\right)+X\left(\frac{\omega}{2}-\pi\right)\right]$$

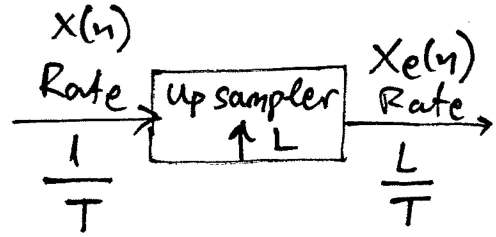




Aliasing would be avoided if we apply a L.P. filter of bandwidth $0<|\omega|<\frac{\pi}{2}$. But then distortion is introduced.

III Interpolation by a factor L (L : integer)

Let $x_e(n) = \begin{cases} x\left(\frac{n}{L}\right), & n = 0, \pm L, \pm 2L \dots \\ 0 & \text{otherwise} \end{cases}$



(L-1 zeros are inserted after each sample of $x(n)$)

In this case $x_e(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-kL) =$

Thus, $X_e(\omega) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-kL)\right] e^{-j\omega n} =$

$= \sum_k x(k) \cdot \sum_n e^{-j\omega n}\,\delta(n-kL) = \sum_k x(k)\, e^{-j\omega kL}$

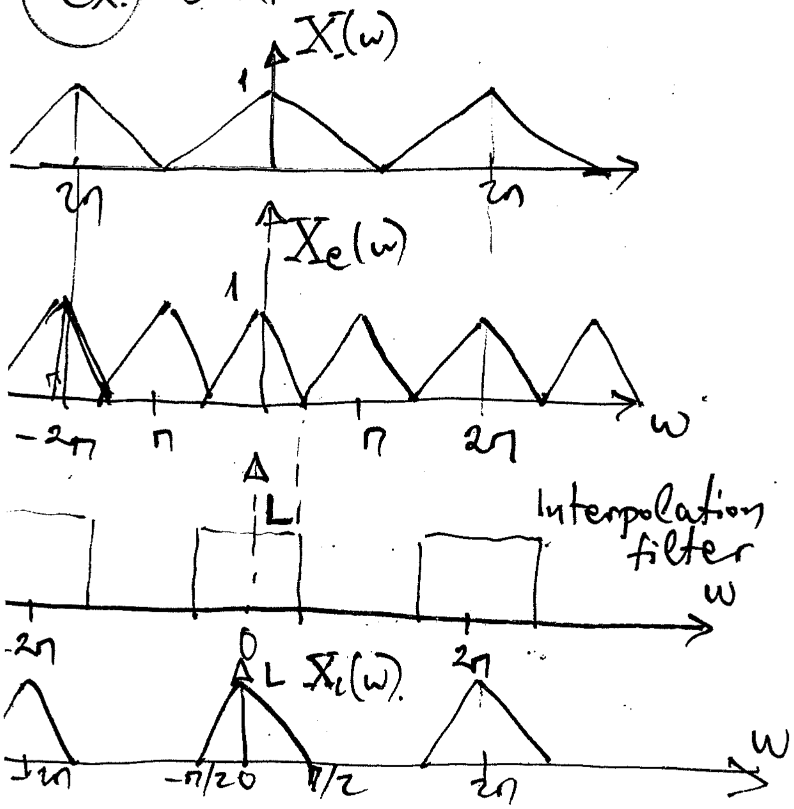
$$\Rightarrow \boxed{X_e(\omega) = X(\omega L)}$$

Compression + Imaging.

in N-DFT terms: $\left[\begin{array}{l} \tilde{x}(n) R_N(n) \xleftrightarrow{N\text{-DFT}} \tilde{X}(k) R_N(k) \\ \tilde{x}_e(n) R_{NL}(n) \xleftrightarrow{NL\text{-DFT}} \tilde{X}(k) R_{NL}(k) \end{array}\right]$

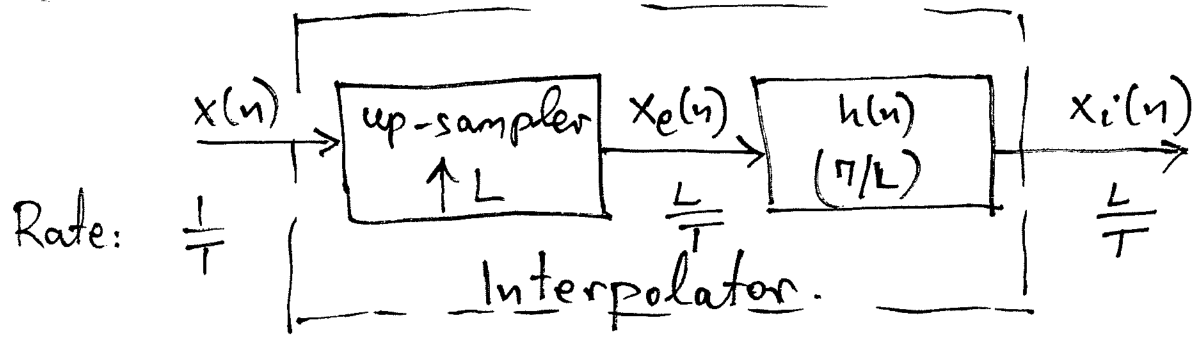
(L-1 images of $X(\omega)$ are inserted into one period of $X_e(\omega)$)

Ex. $L=2$.



If we want to reject the (L-1) images we must filter $x_e(n)$ with a low-pass or in general (band pass) filter of gain L

and bandlimited to $0 \le |\omega| \le \frac{\pi}{L}$. Then, $L-1$ ~~non zero~~ values are interpolated after each value of $x(n)$.



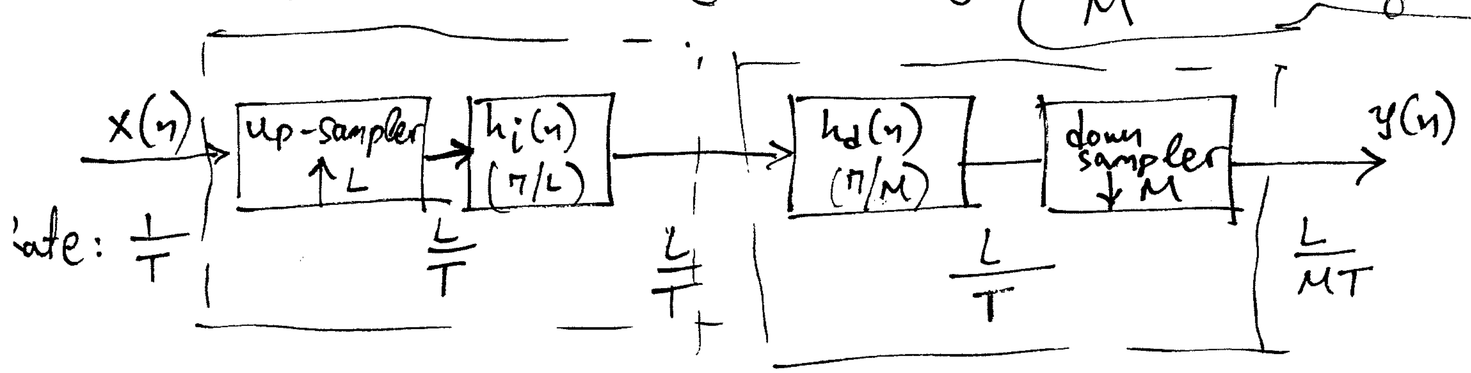
In this case: $x_i(n) = h(n) * x_e(n) = \sum_k h(n-k) x_e(k)$

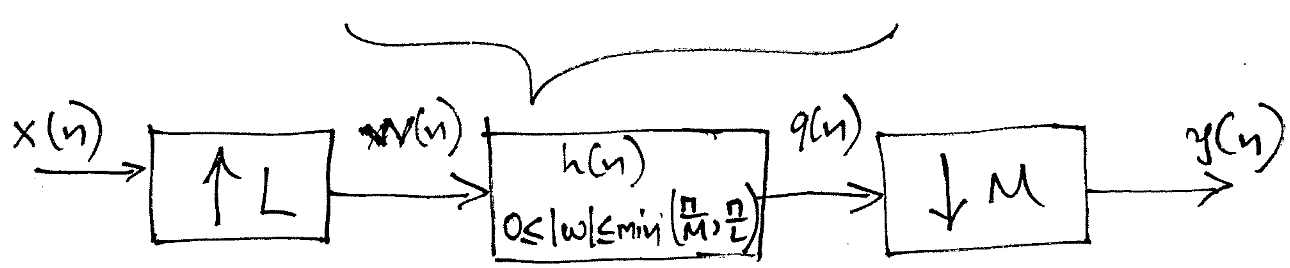
$$= \sum_k h(n-k) \sum_\ell x(\ell)\, \delta(k-\ell L) =$$

$$= \sum_{\ell=-\infty}^{\infty} x(\ell)\, h(n-\ell L)$$

Also, $X_i(\omega) = H(\omega) \cdot X_e(\omega) = H(\omega) \cdot X(L\omega)$

- <u>Change of sampling rate by $\frac{L}{M}$: noninteger</u>





$q(n) = \sum_k x(k)\, h(n-kL) \Rightarrow \boxed{y(n) = \sum_k x(k)\, h(Mn-kL)}$

$Q(\omega) = H(\omega) \cdot X(\omega L) \Rightarrow Y(\omega) = \frac{1}{M} \sum_{i=0}^{M-1} H\left(\frac{\omega - 2\pi i}{M}\right) \cdot X\left(\frac{\omega - 2\pi i}{M} L\right)$

Assuming no aliasing $Y(\omega) = 1/M \cdot H(\omega/M)\, X(\omega L/M)$