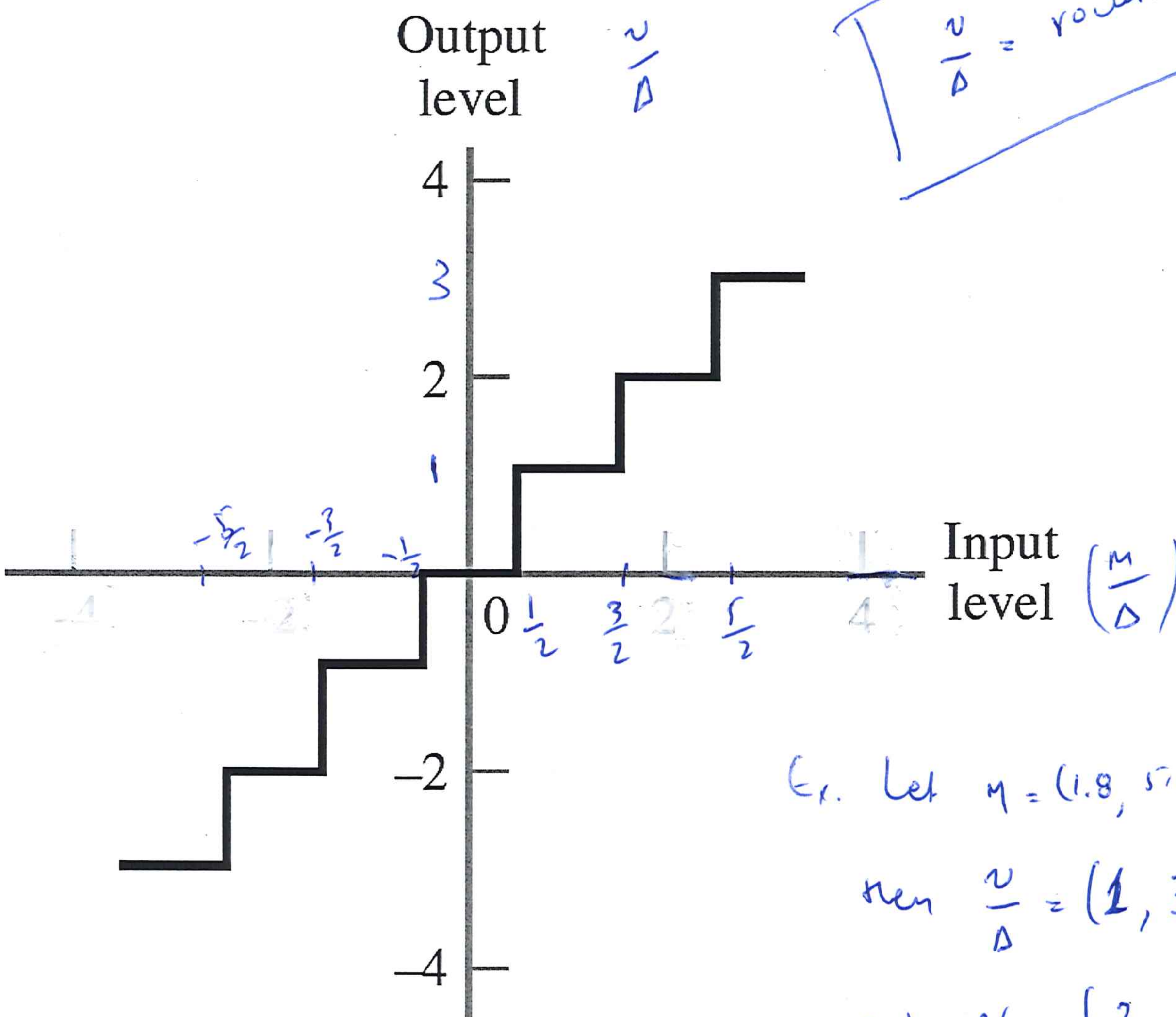
ECE462 – Lecture 4

Uniform Quantization

Midtread Quantizer:



$$\frac{v}{\Delta} = \text{round}\left(\frac{u}{\Delta}\right)$$

Ex. Let $u = (1.8, 5.7, -0.6)$ $\Delta = 2$

then $\frac{v}{\Delta} = (1, 3, 0)$

and $v = (2, 6, 0)$

Quantization and Transform Examples

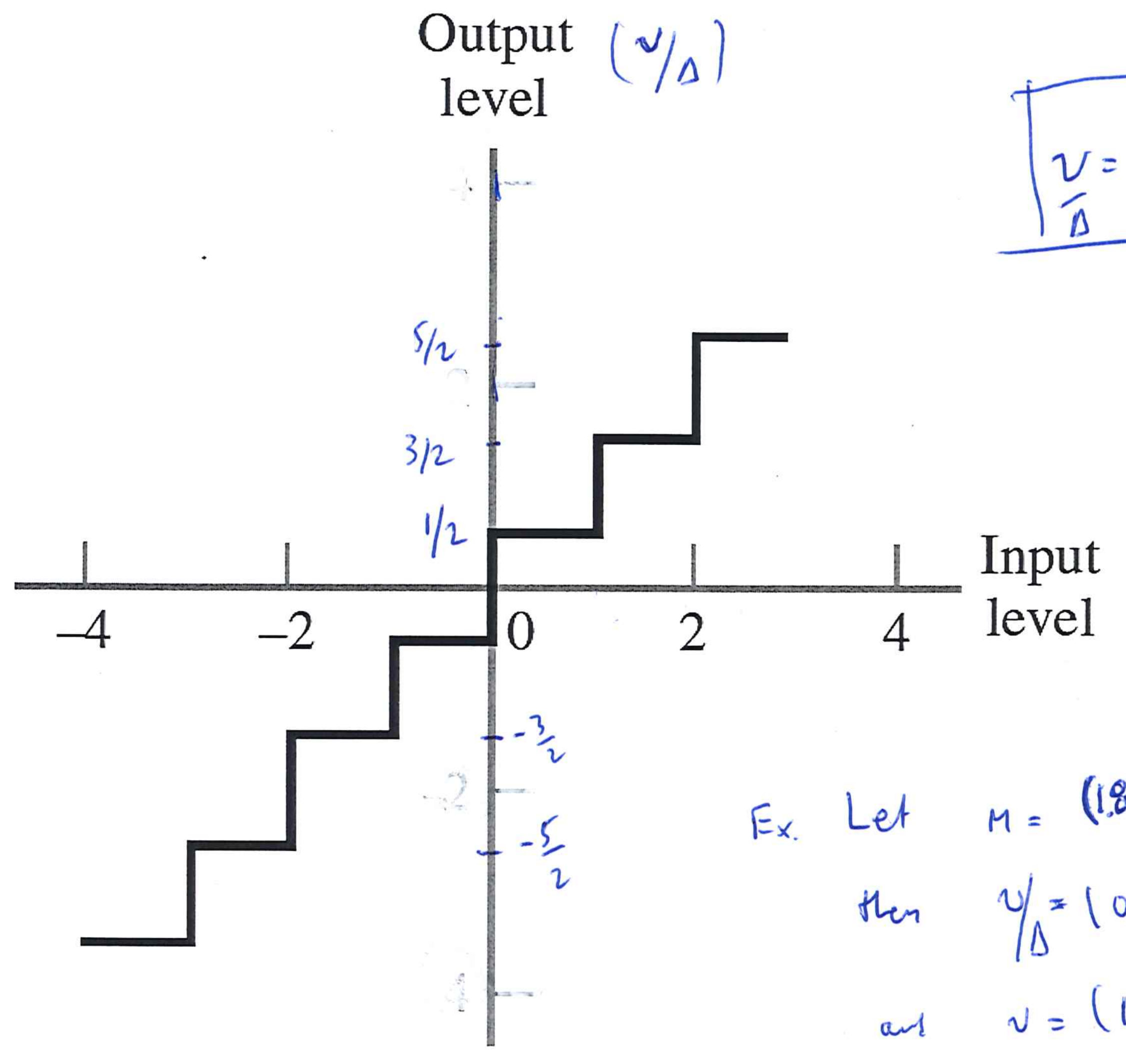
Ex. 1: $x = [1.8, 5.7, -0.6]$, apply midtread uniform quantizer with step size (Δ) 2

$$\begin{aligned} x_q &= \left\lfloor \frac{x}{\Delta} + 0.5 \right\rfloor = \text{round}\left(\frac{x}{\Delta}\right) \\ \therefore\ x_q &= [1, 3, 0] \Rightarrow \text{dequanitze} \Rightarrow \hat{x} = x_q \cdot \Delta = [2, 6, 0] \end{aligned}$$

$$\begin{aligned} \text{MSE} &= \frac{1}{N} \sum_{n=0}^{N-1} (\hat{x}_n - x_n)^2 \\ &= \frac{1}{3}(0.2^2 + 0.3^2 + 0.6^2) \\ &= 0.49/3 \approx 0.163 \end{aligned}$$

Uniform Quantization

Midrise Quantizer: Δ is the step size



$$\frac{v}{\Delta} = \left\lceil \frac{m}{\Delta} \right\rceil - 0.5$$

Ex. Let $m = [1.8, 5.7, -0.6]$, $\Delta = 2$

then $v/\Delta = (0.5, 2.5, -0.5)$

and $v = (1, 5, -1)$

Quantization and Transform Examples

Ex. 2: Apply uniform midrise quantizer: $x_q = \lceil \frac{x}{\Delta} \rceil - 0.5$

$$
\begin{aligned}
\therefore\ x_q &= [0.5, 2.5, -0.5] \\
\hat{x} &= x_q \cdot \Delta = [1, 5, -1] \\
\text{MSE} &= \frac{1}{3}(0.8^2 + 0.7^2 + 0.4^2) \\
&= 0.129/3 \approx 0.43
\end{aligned}
$$

Quantization and Transform Examples

Ex. 3: Assume x is uniformly distributed in the range $[-8, 8]$. Which quantizer should we use if N is even?

$\Rightarrow$ Uniform midrise quantizer, because it allows symmetric matching of positive and negative range with an even number of quantization bins (since zero is *not* a reconstruction value).

If $\Delta = 2$, what is quantization output rate?

$$N = \frac{x_{\max} - x_{\min}}{\Delta} = 8$$
$$R = \log_2 N = 3 \text{ bits}$$

Quantization and Transform Examples

Ex. 4: What is expected distortion (MSE) for Ex. 3?

$$D = E[(x - \hat{x})^2] = \sum_{i=0}^{N-1} \int_{b_i}^{b_{i+1}} (x - \hat{x}_i)^2 f_X(x) dx$$

$\Rightarrow$ uniform quantizer and uniform input:

$$\Rightarrow \ f_X(x) = \frac{1}{16}, \ -8 \leq x \leq 8 \quad \text{(remember, for any pdf } \int_{-\infty}^{\infty} f_X(x) = 1\text{)}$$

Quantization and Transform Examples

Since $\Delta = 2$, the difference between the variable value and the quantizer output reconstruction value, $(x - \hat{x}_i)$, linearly varies between -1 and 1, for each i. Hence, we perform a change of variables that is independent of i.

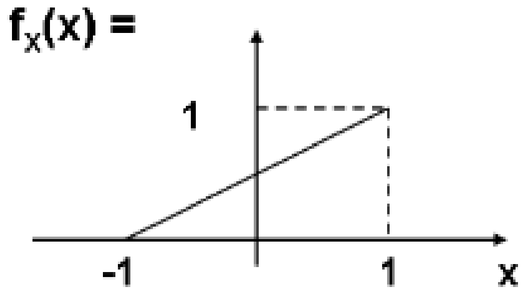
$$\tilde{x} = x - \hat{x} \;\Rightarrow\; -1 \leq \tilde{x} \leq 1$$

Quantization and Transform Examples

$$
\begin{aligned}
\therefore\ D &= \frac{1}{16}\sum_{i=0}^{N-1}\int_{-1}^{1}\tilde{x}^2 dx \\
&= \frac{1}{16}\sum_{i=0}^{N-1}\frac{\tilde{x}^3}{3}\bigg|_{-1}^{1} \\
&= \frac{1}{16}\sum_{i=0}^{N-1}\frac{2}{3} \quad \rightarrow N = 8 \\
&= \frac{8}{16}\cdot\frac{2}{3} = \frac{1}{3} \quad \left(= \frac{\Delta^2}{12}\right)
\end{aligned}
$$

Quantization and Transform Examples

Ex. 5: You have a random variable with the pdf $f_X(x)$ shown below. Calculate the resulting distortion for a 1-bit uniform quantizer with the following parameters: $\Delta = 1$, $x_{\max} = 1$, $x_{\min} = -1$, $N = 2$, $b = [-1, 0, 1]$, $\hat{x} = [-\frac{1}{2}, \frac{1}{2}]$



Quantization and Transform Examples

$$
\begin{aligned}
D &= \sum_{i=0}^{N-1} \int_{b_i}^{b_{i+1}} (x - \hat{x}_i)^2 f_X(x) dx \\
f_X(x) = \frac{x+1}{2} \Rightarrow \quad &= \int_{-1}^{0} \left(x - \left(-\frac{1}{2} \right) \right)^2 \left(\frac{x+1}{2} \right) dx + \int_{0}^{1} \left(x - \frac{1}{2} \right)^2 \left(\frac{x+1}{2} \right) dx \\
&= \int_{-1}^{0} \left(x^2 + x + \frac{1}{4} \right) \left(\frac{x+1}{2} \right) dx + \int_{0}^{1} \left(x^2 - x + \frac{1}{4} \right) \left(\frac{x+1}{2} \right) dx \\
&= \int_{-1}^{0} \frac{x^3 + 2x^2 + \frac{5}{4}x + \frac{1}{4}}{2} dx + \int_{0}^{1} \frac{x^3 - \frac{3}{4}x + \frac{1}{4}}{2} dx \\
&= \frac{1}{2} \left[\left(\frac{x^4}{4} + \frac{2}{3}x^3 + \frac{5}{8}x^2 + \frac{1}{4}x \right) \Bigg|_{-1}^{0} + \left(\frac{x^4}{4} - \frac{3}{8}x^2 + \frac{1}{4}x \right) \Bigg|_{0}^{1} \right] \\
&= \frac{1}{2} \left(\frac{4}{24} \right) = \frac{1}{12}
\end{aligned}
$$

Quantization and Transform Examples

Ex. 6: Calculate SQNR (Signal to Quantization Noise Ratio) for Ex. 5.

$$\text{SQNR} = 10\log_{10}\frac{\sigma_X^2}{\sigma_e^2} \;\Rightarrow\; \sigma_e^2 = \frac{1}{12}$$

$$\begin{aligned}
\mu_X &= \int_{-\infty}^{\infty} x f_X(x)dx \\
&= \int_{-1}^{1} \frac{x(x+1)}{2}dx \\
&= \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Bigg|_{-1}^{1} \\
&= \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}
\end{aligned}$$

Quantization and Transform Examples

$$\begin{aligned}
\sigma_X^2 &= \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx \\
&= \int_{-\infty}^{\infty} x^2 f_X(x)dx - \mu_X^2 \quad \text{(this is a standard identity – try to derive it yourself)} \\
&= \int_{-1}^{1} x^2 \left(\frac{x+1}{2}\right) dx - \left(\frac{1}{3}\right)^2 \\
&= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1} - \frac{1}{9} \\
&= \frac{1}{3} - \frac{1}{9} = \frac{2}{9}
\end{aligned}$$

$$\begin{aligned}
\therefore\ \text{SQNR} &= 10\log_{10}\left(\frac{2}{9}\cdot\frac{12}{1}\right) \\
&= 10\log_{10}\left(\frac{8}{3}\right) = 4.26\ \text{dB}
\end{aligned}$$

Quantization and Transform Examples

Ex. 7: Use the Max-Lloyd algorithm to design a non-uniform quantizer for Ex. 5. Initialize that algorithm using the uniform quantizer previously defined: $b = [-1, 0, 1]$; reconstruction values $\hat{x} = [-\frac{1}{2}, \frac{1}{2}]$. Note: the first and last values in b (b_0 and b_2 in this case) are technically not decision boundaries, but the quantizer end points, $x_{\min}$ and $x_{\max}$ respectively. An N-level quantizer, has $N-1$ decision boundaries and N reconstruction values.

1st Iteration:

$$b_i = \frac{\hat{x}_{i-1} + \hat{x}_i}{2}, \quad \therefore\ b_1 = 0 \quad (\checkmark \text{already the case})$$

$$\hat{x}_i = \frac{\int_{b_i}^{b_{i+1}} x f_X(x) dx}{\int_{b_i}^{b_{i+1}} f_X(x) dx}$$

Quantization and Transform Examples

$$\therefore\ \hat{x}_0 = \frac{\int_{-1}^{0} \frac{x(x+1)}{2} dx}{\int_{-1}^{0} \frac{(x+1)}{2} dx} = \frac{\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{-1}^{0}}{\left(\frac{x^2}{2} + x\right)\Big|_{-1}^{0}} = -\frac{1}{3}$$

$$\hat{x}_1 = \frac{\int_{0}^{1} \frac{x(x+1)}{2} dx}{\int_{0}^{1} \frac{(x+1)}{2} dx} = \frac{\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\Big|_{0}^{1}}{\left(\frac{x^2}{2} + x\right)\Big|_{0}^{1}} = \frac{5}{9}$$

$$\therefore\ \hat{x} = \left[-\frac{1}{3}, \frac{5}{9}\right]$$

Quantization and Transform Examples

Ex. 8: Ex. 8.5, p.221 in text Note: there is an inconsistency in text: they describe the use of the autocorrelation matrix R_X for deriving the KLT, but then use the autocovariance matrix C_X in the example. The KLT does in fact use C_X (though, some people define R_X as we define C_X!), but we often have/assume zero-mean processes ($\mu_X = 0$), therefore $R_X = C_X$.

Ex. 8.5 $\Rightarrow$ Let us assume X is a zero-mean process

We have $N = 4$ sample sequences of length 3 ($X(i)$, $i = \{0, 1, 2\}$):
$x_0 = [4, 4, 5]$, $x_1 = [3, 2, 5]$, $x_2 = [5, 7, 6]$, $x_4 = [6, 7, 7]$

Quantization and Transform Examples

$$R_X = E[XX^T] = E\left(\begin{bmatrix} X(0) \\ X(1) \\ X(2) \end{bmatrix} [X(0)\,X(1)\,X(2)]\right)$$

$$= \begin{bmatrix} E[X(0)X(0)] & E[X(1)X(0)] & E[X(2)X(0)] \\ E[X(0)X(1)] & E[X(1)X(1)] & E[X(2)X(1)] \\ E[X(0)X(2)] & E[X(1)X(2)] & E[X(2)X(2)] \end{bmatrix}$$

Note: diagonal of R_X, $R_X(i,i) = E[X(i)^2] = \sigma^2_{X(i)}$ (when $\mu_X = 0$)

Quantization and Transform Examples

Note: diagonal of R_X, $R_X(i,i) = E[X(i)^2] = \sigma^2_{X(i)}$ (when $\mu_X = 0$)

$$\begin{aligned}
\Rightarrow \text{ Estimate } R_X(i,j) &= \frac{1}{N-1}\sum_{n=0}^{N-1} x_n(i)x_n(j) \\
&= \begin{bmatrix} 28.67 & 33 & 35.67 \\ 33 & 39.33 & 40.33 \\ 35.67 & 40.33 & 45 \end{bmatrix}
\end{aligned}$$

Find the eigenvalues of R_X:

$$\Rightarrow |\lambda I - R_X| = 0 \Rightarrow \lambda = \{111.2052, 1.7485, 0.0463\}$$

where $|\cdot|$ is the determinant operator.

Quantization and Transform Examples

And the eigenvectors:

$$
\begin{aligned}
R_X u = \lambda u \ \Rightarrow \ u_o &= [0.5073, 0.5870, 0.6301] \\
u_1 &= [0.0794, -0.7609, 0.6440] \\
u_2 &= [-0.8581, 0.2766, 0.4326]
\end{aligned}
$$

These eigenvectors from R_X are the bases of the KLT, and thus are assembled as the rows of the transform matrix F:

$$
y = Fx \ \Rightarrow \ F = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}
$$

Quantization and Transform Examples

Remember:

$$
\begin{aligned}
R_Y &= E[YY^T] \\
&= E[FXX^TF^T] \\
&= F\,E[XX^T]\,F^T \\
&= FR_XF^T
\end{aligned}
$$

For KLT:

$$
R_Y = \begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda_{N-1} \end{bmatrix}
$$

$$
\therefore \lambda_i = \sigma^2_{Y(i)} \quad (\text{for } \mu_Y = 0)
$$

Quantization and Transform Examples

Note: λ_0 is large compared to others $\Rightarrow$ good energy compaction.

We can easily calculate

$$\begin{aligned} G_{TC_Y} &= \frac{\frac{1}{N}\sum_{i=0}^{N-1}\sigma^2_{Y(i)}}{\left(\prod_{i=0}^{N-1}\sigma^2_{Y(i)}\right)^{1/N}} = 18.1082 \\ \Rightarrow G_{TC_X} &= 1.0174 \end{aligned}$$

Note: $G_{TC_X} = 1$ if $\sigma^2_{X(i)} = \sigma^2_X$ for all i

Also note that the transform is energy preserving $\sum \sigma^2_X = \sum \sigma^2_Y$ (since it is orthonormal).

Quantization and Transform Examples

Ex. 9: X is a zero-mean process with $R_X(i,j) = 0.95^{|i-j|}$

$$R_X = \begin{bmatrix} 1 & 0.95 & 0.9025 & 0.8574 \\ 0.95 & 1 & 0.95 & 0.9025 \\ 0.9025 & 0.95 & 1 & 0.95 \\ 0.8574 & 0.9025 & 0.95 & 1 \end{bmatrix}$$

$$\Rightarrow \; k = |i-j|; \Rightarrow R_X(k) = 0.95^k = E[X(i)X(i \pm k)]$$

The process is (wide-sense) stationary – the autocorrelation only depends on the distance between the two "points" (k), not the absolute position (i).

Quantization and Transform Examples

Given transform

$$F = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.4 & 0.2 & -0.2 & -0.4 \\ 0.25 & -0.25 & -0.25 & 0.25 \\ 0.2 & -0.4 & 0.4 & -0.2 \end{bmatrix} \text{ (DCT Approximation)}$$

Note F is not orthonormal.

Find $\sigma^2_{Y(i)}$ for $y = Fx$:

$$R_Y \quad = \quad F R_X F^T \rightarrow \text{ but, we only need diagonals – no need to perform entire matrix multiplication}$$

Quantization and Transform Examples

Note F is not orthonormal.

Find $\sigma^2_{Y(i)}$ for $y = Fx$:

$$\begin{aligned}
R_Y &= FR_XF^T \rightarrow \text{ but, we only need diagonals -- no need to perform entire matrix multiplication} \\
\therefore \sigma^2_{Y(0)} &= E[Y_0^2] \\
&= E\left[(0.25 \cdot X(0) + 0.25 \cdot X(1) + 0.25 \cdot X(2) + 0.25 \cdot X(3))^2\right] \\
&= \frac{1}{4}\left[E[X(0)^2] + E[X(0)X(1)] + \ldots\right] \\
\sigma^2_{Y(1)} &= E[Y(1)^2] \\
&= E\left[(0.4 \cdot X(0) + 0.2 \cdot X(1) - 0.2 \cdot X(2) - 0.4 \cdot X(3))^2\right] \\
& \ldots \\
\therefore \sigma^2_{Y(i)} &= [0.939, 0.0648, 0.0128, 0.0122]
\end{aligned}$$