# ECE462 – Lecture 7

Bit assignment and representation in DFT coefficients

#### **Probability Definitions**

 $ECE462\ Multimedia\ Systems$ 

Name	Definition (Continuous r.v.)	Definition (Discrete r.v.)	Estimate from Samples
Mean $(\mu_X)$	$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$	$E[X] = \sum_{k} x p_X(x_k)$	$\overline{x} = \frac{1}{N} \sum_{n=0}^{N-1} x_n$
Variance $(\sigma_X^2)$	$E[(X-\mu_X)^2]=\int_{-\infty}^\infty (x-\mu_X)^2 f_X(x)dx$	$E[(X - \mu_X)^2] = \sum_k (x - \mu_X)^2 p_X(x_k)$	$\frac{1}{N-1}\sum_{n=0}^{N-1}(x_n-\overline{x})^2$
Correlation	$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y) dx dy$	$E[XY] = \sum_{k} \sum_{l} xyp_{X,Y}(x_k, y_l)$	$\frac{1}{N-1}\sum_{n=0}^{N-1}x_ny_n$
$\begin{array}{c} \text{Covariance} \\ (\text{Cov}(X,Y)) \end{array}$	$E[(X-\mu_X)(Y-\mu_Y)]= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f_{X,Y}(x,y)dxdy$	$E[(X - \mu_X)(Y - \mu_Y)] = \sum_k \sum_l (x - \mu_X)(y - \mu_Y)p_{X,Y}(x_k, y_l)$	$\frac{1}{N-1}\sum_{n=0}^{N-1}(x_n-\overline{x})(y_n-\overline{y})$
Autocorrelation $(R_X(i, j))$	$E[X(i)X(j)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X(i),X(j)}(x,y) dx dy$	$E[X(i)X(j)] = \sum_{k} \sum_{l} xyp_{X(i),X(j)}(x_k, y_l)$	$\frac{1}{N-1}\sum_{n=0}^{N-1}x_n(i)x_n(j)$
Autocovariance $(C_X(i, j))$	$E[(X(i) - \mu_{X(i)})(X(j) - \mu_{X(j)})] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{x} \widetilde{y} f_{X(i),X(j)}(x,y) dx dy$ where $\widetilde{x} = (x - \mu_{X(i)})$ and $\widetilde{y} = (y - \mu_{X(j)})$	$E[(X(i) - \mu_{X(i)})(X(j) - \mu_{X(j)})] = \sum_{k} \sum_{l} \widetilde{x} \widetilde{y} p_{X(i),X(j)}(x_k, y_l)$ where $\widetilde{x} = (x - \mu_{X(i)})$ and $\widetilde{y} = (y - \mu_{X(j)})$	$\frac{1}{N-1}\sum_{n=0}^{N-1}(x_n(i)-\overline{x(i)})(x_n(j)-\overline{x(j)})$

See important notes on next page

#### NOTES:

- 1.  $f_X(x)$  is the probability density function (p.d.f.) of continuous random variable (r.v.) X; the probability that X will take on values in the range  $a \le x \le b$  is  $\int_a^b f_X(x) dx$ .
- 2.  $p_X(x_k)$  is the probability mass function (p.m.f.) of discrete r.v. X; the probability that X will take on the value  $x = x_k$  is  $p_X(x_k)$ .
- 3. All definitions assume that the random variables are real-valued.
- 4. For estimates using sample values (last column in the table),  $x_n$  refers to the  $n^{\text{th}}$  sample. For example, if you are estimating the mean of a pixel and you have 3 sample pixel values  $\{5, 9, 2\}$ , you estimate the mean  $\overline{x} = (5 + 9 + 2)/3$ .
- 5. Many of the estimates using samples are scaled by N 1 (rather than N). This makes them unbiased estimates of their respective statistic. In practice you may see either version; in this course either version will be accepted.
- 6. Correlation and covariance is calculated for a pair of random variables, X and Y, where  $f_{X,Y}(x, y)$  is the joint p.d.f. (for continuous random variables), and  $p_{X,Y}(x_k, y_l)$  is the joint p.m.f. (for discrete random variables).
- 7. Autocorrelation and autocovariance are calculated for a random process, X(i), where *i* is the time/position at which the process is sampled. For example, we may consider a row of 5 pixels in an image a random process, where  $0 \le i < 5$  is the position index from left to right. Each X(i) (pixel) can be considered an individual r.v.
- 8. The estimates for autocorrelation and autocovariance require multiple samples for each of the time/positions *i* and *j*. For example, say we have 3 sample sequences,  $x_0 = [1, 4, 7, 10]$ ,  $x_1 = [2, 5, 8, 11]$ , and  $x_3 = [3, 6, 9, 12]$ ; to estimate the autocorrelation at time/position points 0 and 2:

$$R_{\chi}(0,2) \approx \frac{1}{N-1} \sum_{n=0}^{N-1} x_n(0) x_n(2)$$
  
=  $\frac{1}{3-1} (1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9)$   
=  $\frac{1}{2} \cdot 50 = 25$ 

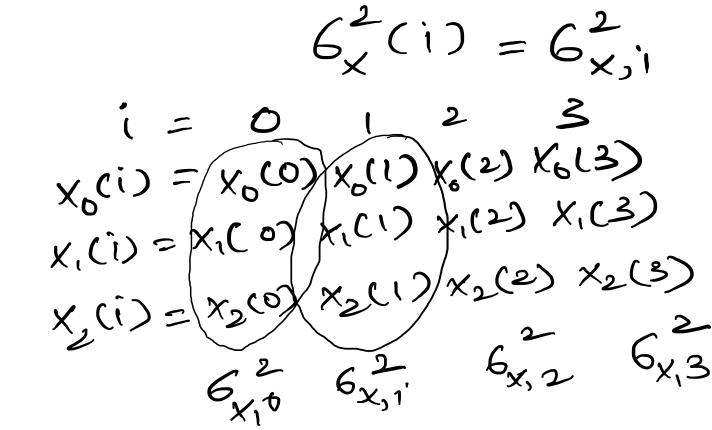
Continuing point 8 the definition of  

$$R_{X}(2,0) = \frac{1}{N-1} \underset{N=0}{\overset{N-1}{\leq}} x_{n}(2) x_{n}(0) = R_{X}(0,2)$$

We can define the Auto correlation Matrix  $R_{x,N}$  of size NxN as the NxN matrix:

 $P_{X,N} = \begin{bmatrix} P_X(0,0) & R_X(0,1) - \dots - R_X(0,N-1) \\ P_X(1,0) & P_X(1,1) - \dots - R_X(1,N-1) \\ \vdots \\ \vdots \\ R_X(N-1,0) & R_X(N-1,1) - \dots - R_X(N-1,N-1) \\ R_X(N-1,0) & R_X(N-1,1) - \dots - R_X(N-1,N-1) \end{bmatrix}$ 

Note that  $R_x(i, i)$ , i.e., the diagonal elements are nothing else but the variance of the process x at time (or position) i.



Where in  $x_j(i)$ , *j* indicates different segment or realization and i indicates sample position. Furthermore, if the process is stationary then all positions have the same variance. In this case:

$$R_{\chi}(i,j) = R_{\chi}(j-i) = R_{\chi}(k)$$

So,

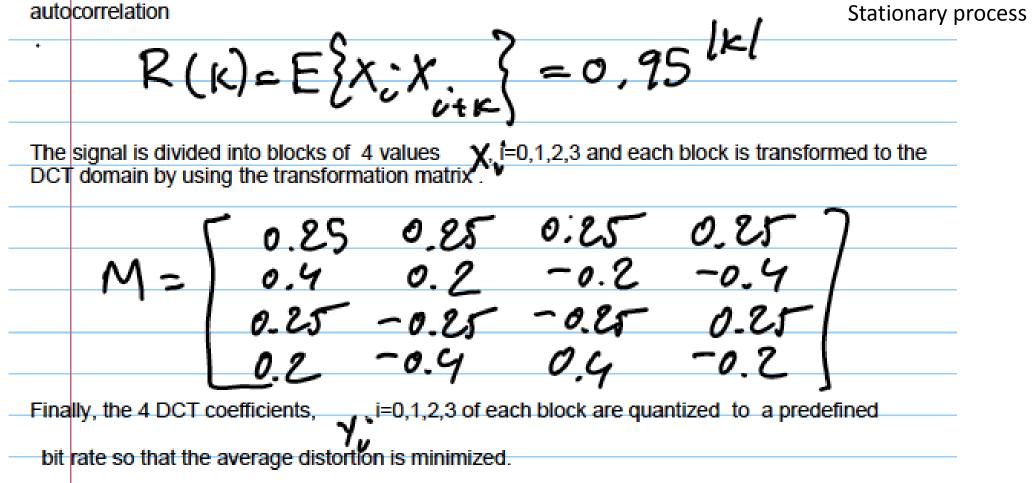
$$\sigma_{x,0}^2 = \sigma_{x,1}^2 = \sigma_{x,2}^2 = \sigma_{x,3}^2$$

And the autocorrelation matrix becomes Toeplitz.

$$R_{X,N} = \begin{bmatrix} R_{x}(0) & R_{x}(1) & \dots & R_{x}(N-1) \\ R_{x}(1) & R_{x}(0) & \dots & R_{x}(N-2) \\ \vdots \\ R_{x}(N-1) & \dots & \dots & R_{x}(0) \end{bmatrix}$$

### Problem

We want to code a signal which is modeled as a zero-mean Gaussian process with



#### Problem

• Calculate the variance of *Y<sub>i</sub>* defined as:

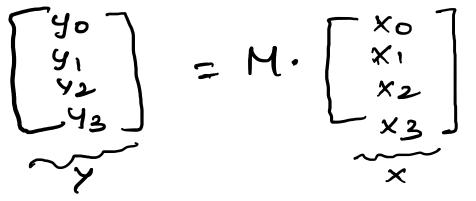
$$6_i^2 = E[y_i^2], i=0,1,2,3$$

• By defining the distortion due to quantization as:

$$D_{i} = 6_{i}^{2} \cdot 2_{j}^{-2R_{i}} = 0_{j} \cdot 2_{j}^{-2R_{i}}$$

Where  $R_i$  are the number of bits assigned to  $y_i$ . Allocate bits to  $y_i$ , i = 0,1,2,3 so that the average rate is 2 bits/sample while the average distortion is minimized.

• Given 4 input sample  $x_i$ , i = 0,1,2,3 the 4-DCT transformation M produces four samples  $y_i$ , i = 0,1,2,3 are as follows:



• Now the variance

$$\mathcal{L}_{y_i}^2 = E\left(\frac{y_i^2}{y_i}\right) - E\left(\frac{y_i^2}{y_i}\right)$$

Since  $x_i$  is zero mean ->  $y_i$  is zero mean.

Then,  $\sigma_{y_i}^2$ , i = 0,1,2,3 are simply the diagonal elements of the matrix:  $\mathcal{E}\left[ \forall . \forall \top \right] = \mathcal{E}\left[ \mathcal{H} \cdot \times \cdot \times^{\top} \mathcal{H}^{\top} \right] = \mathcal{H} \cdot \mathcal{E}\left[ \times \cdot \times^{\top} \mathcal{J} \cdot \mathcal{H}^{\top} \right]$ But,

$$E[x,x^{T}] = \begin{bmatrix} E[x_{0}^{2}] & E[x_{0}x_{1}] & E[x_{0}x_{2}] & E[x_{0}x_{0}] & E[x_{0}x_{0}] & E[x_{0}x_{0}] & E[x_{0}x_{0}]$$

Solution Above is the autocorrelation matrix R. Therefore:  $E[YY^T] = M.R.M^T =$ 

(after some simple calculations)

 $= \begin{bmatrix} 0.94 \\ 0.065 \\ 0.005 \\ 0.0013 \end{bmatrix}$ 

(Note that x is stationary

y is not stationary)

So 
$$\sigma_{y_0}^2$$
=0.94,  $\sigma_{y_1}^2$  = 0.065,  $\sigma_{y_2}^2$  = 0.005,  $\sigma_{y_3}^2$  = 0.013

Thus, if we assign R<sub>i</sub> bits to y<sub>i</sub> then the corresponding distortion will be

$$D_i = 6_{y_i}^2 \cdot 2^{-2k_i}$$

• In order to allocate a total of 8 bits to  $y_0, y_1, y_2, y_3$  (2 bps as the average) we proceed as follows:

 $Jf 8 bits to Y_{0}, Y_{1}.$  i 0 1 2  $fg_{i}^{2} 0.94 0.065 0.005 0.013$   $D_{i} \begin{cases} 1 bit 0.236 0.012 0.001 0.000 \\ 2bits 0.059 0.004 \\ 3bits 0.015 \\ 4.555 0.0367 \\ 7 bit 0.0367 \end{cases}$ 

- Thus, to minimize the distortion assign bits as follows,
- $_{v0}(4 \text{ bits}), y_1(2 \text{ bits}), y_2(1 \text{ bit}), y_3(1 \text{ bit})$

• Then, the average distortion is

$$D_{av} = \frac{1}{4} \left( 0.00367 + 0.004 + 0.001 + 0.003 \right) = 0.0029$$

If on the other hand we have assigned 2 bps for each of  $x_0, x_1, x_2, x_3$  the distortion would be:

$$D_{av} = \sigma_x^2 \cdot 2^{-2.2} = R_x(0) \cdot \frac{1}{16} = \frac{1.1}{16} = 0.0625$$