ECE462 – Lecture 7

Bit assignment and representation in DFT coefficients

Probability Definitions

ECE462 Multimedia Systems

Name	Definition (Continuous r.v.)	Definition (Discrete r.v.)	Estimate from Samples
Mean (μ_X)	$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$	$E[X] = \sum_k x p_X(x_k)$	$\overline{x} = \frac{1}{N} \sum_{n=0}^{N-1} x_n$
Variance (σ_X^2)	$E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x) dx$	$E[(X-\mu_X)^2] = \sum_k (x-\mu_X)^2 p_X(x_k)$	$\frac{1}{N-1} \sum_{n=0}^{N-1} (x_n - \overline{x})^2$
Correlation	$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y) dx dy$	$E[XY] = \sum_k \sum_l xy p_{X,Y}(x_k, y_l)$	$\frac{1}{N-1} \sum_{n=0}^{N-1} x_n y_n$
Covariance ($\mathrm{Cov}(X,Y)$)	$E[(X-\mu_X)(Y-\mu_Y)] =$ $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f_{X,Y}(x,y) dx dy$	$E[(X-\mu_X)(Y-\mu_Y)] =$ $\sum_k \sum_l (x-\mu_X)(y-\mu_Y) p_{X,Y}(x_k, y_l)$	$\frac{1}{N-1} \sum_{n=0}^{N-1} (x_n - \overline{x})(y_n - \overline{y})$
Autocorrelation ($R_X(i,j)$)	$E[X(i)X(j)] =$ $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X(i),X(j)}(x,y) dx dy$	$E[X(i)X(j)] =$ $\sum_k \sum_l xy p_{X(i),X(j)}(x_k, y_l)$	$\frac{1}{N-1} \sum_{n=0}^{N-1} x_n(i) x_n(j)$
Autocovariance ($C_X(i,j)$)	$E[(X(i)-\mu_{X(i)})(X(j)-\mu_{X(j)})] =$ $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{x}\widetilde{y} f_{X(i),X(j)}(x,y) dx dy$ where $\widetilde{x} = (x - \mu_{X(i)})$ and $\widetilde{y} = (y - \mu_{X(j)})$	$E[(X(i)-\mu_{X(i)})(X(j)-\mu_{X(j)})] =$ $\sum_k \sum_l \widetilde{x}\widetilde{y} p_{X(i),X(j)}(x_k, y_l)$ where $\widetilde{x} = (x - \mu_{X(i)})$ and $\widetilde{y} = (y - \mu_{X(j)})$	$\frac{1}{N-1} \sum_{n=0}^{N-1} (x_n(i) - \overline{x(i)})(x_n(j) - \overline{x(j)})$

See important notes on next page

NOTES:

1. $f_X(x)$ is the probability density function (p.d.f.) of continuous random variable (r.v.) X; the probability that X will take on values in the range $a \leq x \leq b$ is $\int_a^b f_X(x)dx$.

2. $p_X(x_k)$ is the probability mass function (p.m.f.) of discrete r.v. X; the probability that X will take on the value $x = x_k$ is $p_X(x_k)$.

3. All definitions assume that the random variables are real-valued.

4. For estimates using sample values (last column in the table), x_n refers to the n^{th} sample. For example, if you are estimating the mean of a pixel and you have 3 sample pixel values $\{5, 9, 2\}$, you estimate the mean $\overline{x} = (5 + 9 + 2)/3$.

5. Many of the estimates using samples are scaled by $N - 1$ (rather than N). This makes them *unbiased* estimates of their respective statistic. In practice you may see either version; in this course either version will be accepted.

6. Correlation and covariance is calculated for a pair of random variables, X and Y, where $f_{X,Y}(x, y)$ is the *joint* p.d.f. (for continuous random variables), and $p_{X,Y}(x_k, y_l)$ is the *joint* p.m.f. (for discrete random variables).

7. Autocorrelation and autocovariance are calculated for a random process, $X(i)$, where i is the time/position at which the process is sampled. For example, we may consider a row of 5 pixels in an image a random process, where $0 \leq i < 5$ is the position index from left to right. Each $X(i)$ (pixel) can be considered an individual r.v.

8. The estimates for autocorrelation and autocovariance require multiple samples for each of the time/positions i and j. For example, say we have 3 sample sequences, $x_0 = [1, 4, 7, 10]$, $x_1 = [2, 5, 8, 11]$, and $x_3 = [3, 6, 9, 12]$; to estimate the autocorrelation at time/position points 0 and 2:

$$\begin{aligned} R_X(0,2) &\approx \frac{1}{N-1}\sum_{n=0}^{N-1} x_n(0)x_n(2) \\ &= \frac{1}{3-1}(1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9) \\ &= \frac{1}{2} \cdot 50 = 25 \end{aligned}$$

Continuing point 8 the definition of

$$R_x(2,0) = \frac{1}{N-1} \sum_{n=0}^{N-1} x_n(2)\, x_n(0) = R_x(0,2)$$

We can define the Auto correlation Matrix $R_{x,N}$ of size NxN as the NxN matrix:

$$R_{x,N} = \begin{bmatrix} R_x(0,0) & R_x(0,1) & \cdots & R_x(0,N-1) \\ R_x(1,0) & R_x(1,1) & \cdots & R_x(1,N-1) \\ \vdots & & & \\ R_x(N-1,0) & R_x(N-1,1) & \cdots & R_x(N-1,N-1) \end{bmatrix}$$

Note that $R_x(i,i)$, i.e., the diagonal elements are nothing else but the variance of the process x at time (or position) i.

$$\sigma_x^2(i) = \sigma_{x,i}^2$$

$$\begin{array}{lcccc}
i = & 0 & 1 & 2 & 3 \\
x_0(i) = & x_0(0) & x_0(1) & x_0(2) & x_0(3) \\
x_1(i) = & x_1(0) & x_1(1) & x_1(2) & x_1(3) \\
x_2(i) = & x_2(0) & x_2(1) & x_2(2) & x_2(3) \\
 & \sigma_{x,0}^2 & \sigma_{x,1}^2 & \sigma_{x,2}^2 & \sigma_{x,3}^2
\end{array}$$

Where in $x_j(\mathrm{i})$,
j $indicates\ different\ segment$ or $realization$ and i indicates sample position.

Furthermore, if the process is stationary then all positions have the same variance. In this case:

$$R_x(i,j) = R_x(j-i) = R_x(k)$$

So,

$$\sigma_{x,0}^2 = \sigma_{x,1}^2 = \sigma_{x,2}^2 = \sigma_{x,3}^2$$

And the autocorrelation matrix becomes Toeplitz.

$$R_{x,N} = \begin{bmatrix} R_x(0) & R_x(1) & \cdots & R_x(N-1) \\ R_x(1) & R_x(0) & \cdots & R_x(N-2) \\ \vdots & & & \\ R_x(N-1) & \cdots & \cdots & R_x(0) \end{bmatrix}$$

Problem

We want to code a signal which is modeled as a zero-mean Gaussian process with autocorrelation

Stationary process

$$R(k) = E\{X_i X_{i+k}\} = 0.95^{|k|}$$

The signal is divided into blocks of 4 values X_i, i=0,1,2,3 and each block is transformed to the DCT domain by using the transformation matrix .

$$M = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.4 & 0.2 & -0.2 & -0.4 \\ 0.25 & -0.25 & -0.25 & 0.25 \\ 0.2 & -0.4 & 0.4 & -0.2 \end{bmatrix}$$

Finally, the 4 DCT coefficients, Y_i, i=0,1,2,3 of each block are quantized to a predefined bit rate so that the average distortion is minimized.

Problem

- Calculate the variance of Y_i defined as:

$$\sigma_i^2 = E[y_i^2], \quad i = 0, 1, 2, 3$$

- By defining the distortion due to quantization as:

$$D_i = \sigma_i^2 \cdot 2^{-2R_i}, \quad i = 0, 1, 2, 3$$

Where R_i are the number of bits assigned to y_i . Allocate bits to $y_i, i = 0,1,2,3$ so that the average rate is 2 bits/sample while the average distortion is minimized.

Solution

- Given 4 input sample $x_i, i = 0,1,2,3$ the 4-DCT transformation M produces four samples $y_i, i = 0,1,2,3$ are as follows:

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{Y} = M \cdot \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X}$$

- Now the variance

$$\sigma_{y_i}^2 = E\left[y_i^2\right] - E^2\left[y_i\right]$$

Solution

Since x_i is zero mean -> y_i is zero mean.

Then, $\sigma_{y_i}^2, i = 0,1,2,3$ are simply the diagonal elements of the matrix:

$$E[Y \cdot Y^T] = E[M \cdot X \cdot X^T \cdot M^T] = M \cdot E[X \cdot X^T] \cdot M^T$$

But,

$$E[X \cdot X^T] = \begin{bmatrix} E[x_0^2] & E[x_0 x_1] & E[x_0 x_2] & E[x_0 x_3] \\ E[x_1 x_0] & E[x_1^2] & \cdots & \\ \cdots & & & \end{bmatrix} = \begin{bmatrix} R_x(0) & R_x(1) & R_x(2) & R_x(3) \\ R_x(-1) & R_x(0) & R_x(1) & R_x(2) \\ R_x(-2) & R_x(-1) & R_x(0) & R_x(1) \\ R_x(-3) & R_x(-2) & R_x(-1) & R_x(0) \end{bmatrix}$$

Solution

Above is the autocorrelation matrix R.

Therefore: $E[YY^T] = M.R.M^T =$

(after some simple calculations)

$$= \begin{bmatrix} 0.94 & & & \\ & 0.065 & & \\ & & 0.005 & \\ & & & 0.0013 \end{bmatrix}$$

(Note that x is stationary

y is not stationary)

So $\sigma_{y_0}^2$=0.94, $\sigma_{y_1}^2 = 0.065, \sigma_{y_2}^2 = 0.005, \sigma_{y_3}^2 = 0.013$

Thus, if we assign R_i bits to y_i then the corresponding distortion will be

$$D_i = \sigma_{y_i}^2 \cdot 2^{-2R_i}$$

Solution

- In order to allocate a total of 8 bits to y_0, y_1, y_2, y_3 (2 bps as the average) we proceed as follows:

	i	0	1	2	3
	$\sigma_{y_i}^2$	0.94	0.065	0.005	0.013
D_i	1 bit	0.236	0.012	0.001	0.003
	2 bits	0.059	0.004		
	3 bits	0.015			
	4 bits	0.0367			

- Thus, to minimize the distortion assign bits as follows,
- y_0(4 bits), y_1(2 bits), y_2(1 bit), y_3(1 bit)

Solution

- Then, the average distortion is

$$D_{av}=\frac{1}{4}(0.00367+0.004+0.001+0.003)=0.0029$$

If on the other hand we have assigned 2 bps for each of x_0,x_1,x_2,x_3 the distortion would be:

$$D_{av}=\sigma_x^2.\,2^{-2.2}=R_x(0).\frac{1}{16}=\frac{1.1}{16}=0.0625$$