Chapter 3 Questions

Question 3.1

Based on the nature of values that each random variable can take, we can have the following classifications:

X: Discrete; since X is essentially count data
Y: Continuous; since Y is measured data
M: Continuous; since M is measured data
N: Discrete; since N is count data
P: Discrete; since P is count data
Q: Continuous; since Q is measured data

Question 3.3

The results could be summarized into the table below:

<table>
<thead>
<tr>
<th>Sample point</th>
<th>HHH</th>
<th>HHT</th>
<th>HTH</th>
<th>HTT</th>
<th>THH</th>
<th>THT</th>
<th>TTH</th>
<th>TTT</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>3-0 = 3</td>
<td>2-1 = 1</td>
<td>2-1 = 1</td>
<td>1-2 = -1</td>
<td>2-1 = 1</td>
<td>1-2 = -1</td>
<td>1-2 = -1</td>
<td>0-3 = -3</td>
</tr>
</tbody>
</table>

Question 3.5

The principal rule for solving this question is that all probabilities for all sample points within the sample space should add up to 1.

Part (a)

We desire:

\[ \sum_{x=0}^{3} c(x^2 + 4) = 1 \]

So:

\[ c \times [(0 + 4) + (1 + 4) + (4 + 4) + (9 + 4)] = 1 \]

\[ c = \frac{1}{30} \]

Part (b)

Following the similar manner of computation:

\[ C(\binom{2}{0} \binom{3}{3} + \binom{2}{1} \binom{3}{2} + \binom{2}{2} \binom{3}{1}) = 1 \]
\[ C = \frac{1}{1 + 2 \times 3 + 1 \times 3} = \frac{1}{10} \]

**Question 3.8**

We first compute the probability associated to each sample point, then add up the corresponding sample points to obtain probability associated with each value of \( W \).

\[ P(W=3) = P(TTT) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27} \]

\[ P(W=1) = P(HTT) + P(THT) + P(TTH) = 3 \times \left( \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} \right) = \frac{2}{9} \]

\[ P(W=3) = P(HHT) + P(HTH) + P(THH) = 3 \times \left( \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} \right) = \frac{4}{9} \]

\[ P(W=3) = P(HHH) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27} \]

Verify they sum up to one:

\[ \frac{1}{27} + \frac{2}{9} + \frac{4}{9} + \frac{8}{27} = 1 \]

Thus, above describes a valid probability distribution on discrete random variable \( W \).

**Question 3.9**

Part (a)

Integrate over the value of \( X \) to obtain the area under this part of the density function:

\[ P(0 < X < 1) = \int_{0}^{1} \frac{12(x+2)}{5} \, dx \]

\[ = \left[ \frac{x^2 + 4x}{5} \right]_{0}^{1} \]

\[ = \frac{1 + 4}{5} - 0 \]

\[ = 1 \]

Part (b)

Still use integration over density function to compute the probability:

\[ P(1/4 < X < 1/2) = \int_{1/4}^{1/2} \frac{2(x+2)}{5} \, dx \]

\[ = \left[ \frac{x^2 + 4x}{5} \right]_{1/4}^{1/2} \]

\[ = \frac{(1/4 + 2)/5 - (1/16 + 1)/5}{19/80} \]

\[ = \frac{19}{80} \]
Question 3.11

First compute probability distribution of discrete random variable X:

Note that the values X could take on are: 0, 1, 2.

\[ P(X=0) = \binom{5}{3}/\binom{7}{3} = 10/35 = 2/7 \]

\[ P(X=1) = \binom{5}{2}/\binom{7}{3} = (10*2)/35 = 4/7 \]

\[ P(X=2) = \binom{5}{1}/\binom{7}{3} = (5*1)/35 = 1/7 \]

Note that they sum up to 1, thus the above distribution is a valid probability distribution.

The probability histogram is as following:

![Probability Histogram](image)

Question 3.13

To obtain the cumulative distribution function, we just need to performing a cumulative running sum of the probability mass function on X.

Thus, the cumulative distribution function is shown as following:

<table>
<thead>
<tr>
<th>X</th>
<th>x &lt; 0</th>
<th>0 ≤ x &lt; 1</th>
<th>1 ≤ x &lt; 2</th>
<th>2 ≤ x &lt; 3</th>
<th>3 ≤ x &lt; 4</th>
<th>x ≥ 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>F(x)</td>
<td>0</td>
<td>0.41</td>
<td>0.41+0.37 =0.78</td>
<td>0.78+0.16=0.94</td>
<td>0.94+0.05=0.99</td>
<td>0.99+0.01 = 1</td>
</tr>
</tbody>
</table>
Note the ending value for the cumulative function always equals to 1.

Question 3.14

Part (a)

We just need to directly plot in the value of x into F(x):

With 12 minutes = 0.2 hours, we have:

\[ F(x = 0.5) = 1 - e^{-8 \times 0.2} = 0.7981 \]

Part (b)

First, compute probability density function:

\[ f(x) = F'(x) = 8e^{-8x} \]

Then, compute the probability using integration over the density function:

\[ p = \int_{0}^{0.2} f(x) dx \]

\[ = 8 \int_{0}^{0.2} e^{-8x} dx \]

\[ = 8 \times \frac{-1}{8} [e^{-8x}]_{0}^{0.2} \]

\[ = 1 - e^{-8 \times 0.2} \]

\[ = 0.7981 \]

Question 3.15

First, find the cumulative distribution function F(x):

Note that from question 3.11:

P(X=0) = 2/7

P(X=1) = 4/7

P(X=2) = 1/7

Then the cumulative function for F(x) is shown as following:

<table>
<thead>
<tr>
<th>x</th>
<th>x &lt; 0</th>
<th>0 ≤ x &lt; 1</th>
<th>1 ≤ x &lt; 2</th>
<th>x ≥ 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>F(x)</td>
<td>0</td>
<td>2/7</td>
<td>2/7 + 4/7 = 6/7</td>
<td>6/7 + 1/7 = 1</td>
</tr>
</tbody>
</table>

Then use F(x) for asked quantities.
Part (a)

\[ P(X = 1) = F(X = 1) - F(X = 0) = \frac{6}{7} - \frac{2}{7} = \frac{4}{7} \] (since there is no probability mass within (0,1))

Part (b)

\[ P(0 < X \leq 2) = P(X \leq 2) - P(X \leq 0) = F(X = 2) - F(X = 0) = 1 - \frac{2}{7} = \frac{5}{7} \]

Question 3.16

The graph is shown as following:

![Graph Image]

Question 3.29

Part (a)

For this density function to be valid, it has to satisfy two conditions:

1. \( f(x) \geq 0 \ \forall x \)
2. $\int_{-\infty}^{+\infty} f(x)dx = 1$

Condition 1 is easily verified as exponential function never takes on negative values.

For condition 2 verification:

$$
\int_{-\infty}^{+\infty} f(x)dx = \int_{1}^{+\infty} 3x^{-4}dx
= -[x^{-3}]_{1}^{+\infty}
= 1
$$

Thus this density function is indeed valid.

Part (b)

F(x) is computed by the integration of f(x):

$$
F(X) = \int_{\infty}^{X} f(a)da
= \int_{1}^{X} 3a^{-4}da
= -[a^{-3}]_{1}^{X}
= 1 - x^{-3} \ (for \ x \geq 1)
$$

For x<1, as f(x)=0 everywhere, F(x)=0.

Part (c)

The probability can be computed using compliment:

$$
P = 1-F(4)
= 1 - \left(1 - 4^{-3}\right)
= 1/64
$$

Question 3.36

Given the questions asked are about probabilities of certain range, compute the cumulative distribution function first:

$$
F(x) = \int_{\infty}^{x} f(a)da
= \int_{0}^{x} 2(1-a)da
$$
\[ [-a^2 + 2a]_0^x = -x^2 + 2x \text{ (if } 0 \leq x \leq 1) \]

If \( x < 0 \): trivially \( F(x) = 0 \)

If \( x > 1 \): \( F(x) = 1 \)

Part (a)

Use cumulative function:

\[ P(X \leq 1/3) = F(x = 1/3) = -\left(\frac{1}{3}\right)^2 + \frac{2}{3} = \frac{5}{9} \]

Part (b)

Use compliment for computing the desired probability, with cumulative function:

\[ P = 1 - P(x = 0.5) \]
\[ = 1 - F(x = 0.5) \]
\[ = 1 - (-0.5^2 + 2 \times 0.5) \]
\[ = 0.25 \]

Part (c)

\[ P(x < 0.75 | x \geq 0.5) = \frac{P(0.5 \leq x < 0.75)}{P(x \geq 0.5)} \]
\[ = \frac{F(x = 0.75) - F(x = 0.5)}{1 - F(x = 0.5)} \]
\[ = \frac{-0.75^2 + 2 \times 0.75 - (-0.5^2 + 2 \times 0.5)}{1 - (-0.5^2 + 2 \times 0.5)} \]
\[ = \frac{0.9375 - 0.75}{0.25} \]
\[ = 0.75 \]
Problem 3.38

Part (a)

\[ p(X \leq 2, Y = 1) = p(X = 0, Y = 1) + p(X = 1, Y = 1) + p(X = 2, Y = 1) = f(0,1) + f(1,1) + f(2,1) \]
\[ = \frac{0 + 1}{30} + \frac{1 + 1}{30} + \frac{2 + 1}{30} = \frac{1}{5} \]

Part (b)

\[ p(X > 2, Y \leq 1) = p(X = 3, Y = 0) + p(X = 3, Y = 1) = f(3,0) + f(3,1) \]
\[ = \frac{3 + 0}{30} + \frac{3 + 1}{30} = \frac{7}{30} \]

Part (c)

\[ p(X > Y) = f(1,0) + f(2,0) + f(2,1) + f(3,0) + f(3,1) + f(3,2) \]
\[ = \frac{(1 + 0) + (2 + 0) + (2 + 1) + (3 + 0) + (3 + 1) + (3 + 2)}{30} \]
\[ = \frac{3}{5} \]

Part (d)

\[ p(X + Y = 4) = f(2,2) + f(3,1) \]
\[ = \frac{(2 + 2) + (3 + 1)}{30} \]
\[ = \frac{4}{15} \]
Part (a)
Integrate the joint density function over all values of $Y$ to get marginal density of $X$:

$$g(x) = \int_0^1 f(x,y) dy$$

$$= \int_0^1 \frac{2}{3} (x + 2y) dy$$

$$= \frac{2}{3} [xy + y^2]_0^1$$

$$= \frac{2x + 2}{3}$$

Part (b)
Integrate the joint density function over all values of $X$ to get marginal density of $Y$:

$$h(y) = \int_0^1 f(x,y) dx$$

$$= \int_0^1 \frac{2}{3} (x + 2y) dx$$

$$= \frac{2}{3} \left[ \frac{x^2}{2} + 2xy \right]_0^1$$

$$= \frac{4y + 1}{3}$$

Part (c)
This event corresponds to the condition on random variable: $X \leq 0.5$.

As only the drive-through facility is in the picture of the problem, we only need to consider the marginal density function $g(x)$.

Integrate over the density function to get the desired event probability:

$$p(X \leq 0.5) = \int_0^{0.5} g(x) dx$$

$$= \int_0^{0.5} \frac{2x + 2}{3} dx$$
\[
\frac{x^2 + 2x}{3} \bigg|_{0}^{0.5} = \frac{1.25}{3} = \frac{5}{12}
\]

Problem 3.56

Part (a)

Note that for values of \(f(x,y)\), \(Y\)'s range is actually defined based on \(x\), we can have the following observation:

Given a fixed value of \(Y = 0.5\):

Case 1: \(x = 0.3\). Then \(y < 1-x\), and thus \(f(0.3, 0.5) = 6 \times 0.3 = 1.8\) (Note that since it’s density function, so a value greater than 1 is possible).

Case 2: \(x = 0.7\). Then \(y > 1-x\), and thus \(f(0.7, 0.5) = 0\)

Thus we have \(f(0.3, 0.5) \neq f(0.7, 0.5)\), meaning the probability density for \(y = 0.5\) is dependent on \(x\). Thus, \(X\) and \(Y\) are not independent.

Part (b)

Note that for \(Y = 0.5\), when \(X \geq 0.5\), \(y < 1-x\) doesn’t hold and leads to probability density 0. Thus, we only need to consider domains of \(X\) where the density is non-zero.

\[
P(X > 0.3 \mid Y = 0.5)
= \int_{0.3}^{1} f(x, 0.5) dx
= \int_{0.3}^{0.5} f(x, 0.5) dx
= \int_{0.3}^{0.5} 6x dx
= [3x^2]_{0.3}^{0.5}
= 0.48
\]
Problem 3.57

Part (a)

The requirement for probability density function to be valid is that it would be integrated to 1 over all domains.

\[
\int_0^1 \int_0^1 \int_0^1 f(x, y, z) \, dx \, dy \, dz
\]

\[
= \int_0^1 \int_0^1 \int_0^1 kxy^2z \, dx \, dy \, dz
\]

\[
= \int_0^1 \int_0^1 \frac{kxy^2}{2} \, dy \, dz
\]

\[
= \int_0^1 \frac{kz}{6} \, dz
\]

\[
= \frac{k}{3}
\]

\[
= 1 \text{ (by definition)}
\]

Thus, \( k = 3 \).

Part (b)

Integrate the corresponding region with respect to each of the variable, we have:

\[
p(X < \frac{1}{4}, Y > \frac{1}{2}, 1 < Z < 2)
\]

\[
= \int_1^2 \int_{\frac{1}{2}}^{\frac{1}{4}} \int_0^1 f(x, y, z) \, dx \, dy \, dz
\]

\[
= \int_1^2 \int_{\frac{1}{2}}^{\frac{1}{4}} \int_0^1 kxy^2z \, dx \, dy \, dz
\]

\[
= \int_1^2 \int_{\frac{1}{2}}^{\frac{1}{4}} \frac{kxy^2}{32} \, dy \, dz
\]

\[
= \int_1^2 \frac{7kz}{768} \, dz
\]

\[
= \left[ \frac{7kz^2}{1536} \right]_1^2
\]
\[
\begin{align*}
&= \frac{63}{1536} \\
&= \frac{21}{512}
\end{align*}
\]