Problem 4.6

Let random variable "X" denote earning, then according to expectation definition:

\[ E(X) = \sum_{x \in X} x p(x) \]

\[ = \frac{1}{12} \times 7 + \frac{1}{12} \times 9 + \frac{1}{4} \times 11 + \frac{1}{4} \times 13 + \frac{1}{6} \times 15 + \frac{1}{6} \times 17 \]

\[ = \frac{38}{3} \]

Problem 4.7

According to definition of expectation, the expected gain is computed as following:

\[ E(X) = \sum_{x \in X} x p(x) \]

\[ = 4000 \times 0.3 + (-1000) \times 0.7 \]

\[ = 500 \]

Problem 4.12

Taking expectation over continuous probability density function:

\[ E(X) = \int_{-\infty}^{+\infty} x f(x) dx \]

\[ = \int_{0}^{1} x \times 2(1-x) dx \]

\[ = \left[ x^2 - \frac{2x^3}{3} \right]_{0}^{1} \]

\[ = \frac{1}{3} \]

Thus, the profit in money figure is \( \frac{1}{3} \times 5000 = 1666.67 \).
4.17 Expected value of the random variable $g(X)$ is:

\[ \mu_{g(X)} = \sum_{x \in X} g(x)f(x) \]

\[ = (2 \times (-3) + 1)^2 \times \frac{1}{6} + (2 \times 6 + 1)^2 \times \frac{1}{2} + (2 \times 9 + 1)^2 \times \frac{1}{3} \]

\[ = 209 \]

4.18 Expected value of the random variable $g(X)$ is:

\[ E[g(X)] = \sum_{x \in X} g(x)f(x) \]

\[ = \sum_{x \in X} x^2f(x) \]

\[ = \sum_{x \in X} x^2 \left(\frac{3}{x}\right) \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{3-x} \]

\[ = 0^2 \left(\frac{3}{0}\right) \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{3-0} + 1^2 \left(\frac{3}{1}\right) \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{3-1} + 2^2 \left(\frac{3}{2}\right) \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{3-2} + 3^2 \left(\frac{3}{3}\right) \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{3-3} \]

\[ = 0 + 1 \times 3 \times \frac{1}{4} \times \frac{9}{16} + 4 \times 3 \times \frac{1}{16} \times \frac{3}{4} + 9 \times 1 \times \frac{1}{64} \times 1 \]

\[ = \frac{9}{8} \]

4.21 The average profit is computed by expectation of the random variable $g(X) = x^2$:

\[ E[g(X)] = \int_0^1 x^2f(x)dx \]

\[ = \int_0^1 x^2 \times 2(1 - x)dx \]

\[ = \left[\frac{2}{3}x^3 - \frac{1}{2}x^4\right]_0^1 = \frac{1}{6} \]

Thus, the profit in money figure is $\frac{1}{6} \times $5000 = $833.33.
Problem 4.23

Part (a)

\[
\mathbb{E}(g(x, y)) = \sum_{x \in X, y \in Y} g(x, y) \ p(x, y)
\]

\[
= \sum_{x \in X, y \in Y} x^2 y^2 \ p(x, y)
\]

\[
= 2 \times 1^2 \times 0.1 + 4 \times 1^2 \times 0.15 + 2 \times 3^2 \times 0.2 + 4 \times 3^2 \times 0.3
\]

\[
+ 2 \times 5^2 \times 0.1 + 4 \times 5^2 \times 0.15
\]

\[
= 35.2
\]

Part (b)

\[
M_x = \sum_{x \in X} x \ p(x) \quad M_y = \sum_{y \in Y} y \ p(y)
\]

We need the marginal probabilities \( p(x) \) and \( p(y) \).

Note: \( p(x) = \sum_{y \in Y} p(x, y) \) and \( p(y) = \sum_{x \in X} p(x, y) \).

Thus we have:

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<thead>
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<tr>
<td></td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>( p(x) )</td>
<td>0.4</td>
<td>0.6</td>
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<tr>
<th></th>
<th>1</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(y) )</td>
<td>0.25</td>
<td>0.5</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Then, we can proceed to compute:

\[
M_x = 2 \times 0.4 + 4 \times 0.6 = 3.2
\]

\[
M_y = 1 \times 0.25 + 3 \times 0.5 + 5 \times 0.25 = 3.0
\]

Problem 4.35

According to the Theorem 4.2, we have:

\[
6^2 = \mathbb{E}(X^2) - M^2
\]

\[
M = \sum_{x \in X} x \ f(x) = 2 \times 0.01 + 3 \times 0.025 + 4 \times 0.04 + 5 \times 0.05 + 6 \times 0.04 = 4.11
\]

\[
E(X^2) = \sum_{x \in X} x^2 \ f(x) = 2^2 \times 0.01 + 3^2 \times 0.025 + 4^2 \times 0.04 + 5^2 \times 0.05 + 6^2 \times 0.04 = 17.63
\]

Thus \( 6^2 = 17.63 - 4.11 = 13.52 \)
Problem 4.41

To compute standard deviation:

$$
\sigma = \sqrt{E[(g(x))^2] - \mu_{g(x)}^2}
$$

From Problem 4.17, we have obtained:

$$
\mu_{g(x)} = 2.9
$$

To compute $$E[g(x)^2]$$:

$$
E[g(x)^2] = \sum_{x \in \mathbb{R}} g(x)^2 f(x)
$$

$$
= \sum_{x \in \mathbb{R}} (2x+1)^4 f(x)
$$

$$
= \left[2(x-3)+1\right]^4 + \left[2(x+6)+1\right]^4 + \left[2(x+9)+1\right]^4
$$

$$
= 578.25
$$

Therefore:

$$
6 = \sqrt{578.25 - 2.9^2} = \sqrt{141.44} = 8.0221
$$

Problem 4.50

Variance:

$$
\sigma_x^2 = E(x^2) - \mu^2
$$

$$
\mu = \int_0^1 f(x) \cdot x \, dx
$$

$$
E(x^2) = \int_0^1 f(x) \cdot x^2 \, dx
$$

$$
= \int_0^1 2(1-x) \cdot x \, dx
$$

$$
= \int_0^1 2x - \frac{2}{3} x^3 \, dx
$$

$$
= \left[ \frac{2}{3} x^2 - \frac{1}{6} x^4 \right]_0^1
$$

$$
= \frac{1}{3} - \frac{1}{6}
$$

Thus:

$$
\sigma_x^2 = \frac{1}{6} - \left(\frac{1}{2}\right)^2 = \frac{1}{8}
$$

Standard deviation:

$$
\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\frac{1}{8}} = \frac{\sqrt{2}}{6}
$$
Problem 4.58

According to Theorem 4.6, we have:

\[ E(g(x) + h(x)) = E(g(x)) + E(h(x)) \]

Thus, computing expectation of \( Y \) as summary over two functions of \( X \):

\[ E(Y) = E(60X^2 + 39X) = E(60X^2) + E(39X) \]

\[ = 60E(X^2) + 39E(X) \]

\[ E(X^2) = \int_0^1 x^2 f(x) \, dx \]

\[ = \int_0^1 x^2 \cdot x \, dx + \int_1^2 x^2 (2-x) \, dx \]

\[ = \left[ \frac{x^4}{4} \right]_0^1 + \left[ \frac{2}{3}x^3 - \frac{1}{4}x^4 \right]_1^2 \]

\[ = \frac{1}{4} + \frac{11}{12} = \frac{7}{6} \]

\[ E(X) = \int_0^2 x f(x) \, dx \]

\[ = \int_0^1 x \cdot x \, dx + \int_1^2 x (2-x) \, dx \]

\[ = \left[ \frac{x^2}{2} \right]_0^1 + \left[ x^2 - \frac{x^3}{3} \right]_1^2 \]

\[ = \frac{1}{2} + \frac{2}{3} = 1 \]

Thus: \( E(Y) = 60 \cdot \frac{7}{6} + 39 \cdot 1 = 109 \)

---

Problem 4.60

Part (a) \( E(2X - 3Y) \)

\[ E(2X - 3Y) = E(2X) - E(3Y) \]

\[ = 2E(X) - 3E(Y) \]

Marginal distribution:

<table>
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<tr>
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</tr>
</tbody>
</table>

\[ E(X) = \sum_{x} x \cdot p(x) = 2 \cdot 0.4 + 4 \cdot 0.6 = 3.2 \]

\[ E(Y) = \sum_{y} y \cdot p(y) = 1 \cdot 0.25 + 3 \cdot 0.5 + 5 \cdot 0.25 = 3.0 \]

\[ = 2 \cdot 3.2 - 3 \cdot 3.0 = -2.6 \]
Part (b) since \( X \) and \( Y \) are independent, we have:

\[
E(XY) = E(X)E(Y)
\]

From part (a), we have \( E(X) = 3.2 \) and \( E(Y) = 3.5 \).

So:

\[
E(XY) = 3.2 \times 3.5 = 9.6
\]

Problem 4.65

Part (a) Note that red die toss event is independent of the green die toss event. Thus \( X \) and \( Y \) are independent random variables.

\[
E(X + Y) = E(X) + E(Y)
\]

For a die toss, there is \( \frac{1}{6} \) probability for outcome of \( 1 \) to \( 6 \), thus

\[
E(X) = E(Y) = \frac{1}{6} (1+2+3+4+5+6) = 3.5
\]

So:

\[
E(X + Y) = 3.5 + 3.5 = 7.0
\]

Part (b) \( E(X - Y) = E(X) - E(Y) = 3.5 - 3.5 = 0 \)

Part (c) Because \( X, Y \) are independent random variables:

\[
E(XY) = E(X)E(Y) = 3.5 \times 3.5 = 12.25
\]

Problem 4.66

Similar to Problem 4.65, \( X \) and \( Y \) are independent.

Part (a) \( 6^2(2X - Y) = 2^2 6^2X + (-1)^2 6^2Y = 4 \cdot 6^2X + 6^2Y \)

\[
6X^2 = E(X^2) - M_X = \frac{1}{6} [1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2] - [\frac{1}{6} (1+2+3+4+5+6)]^2
\]

\[
= \frac{91}{6} - \frac{49}{4} = \frac{35}{12}
\]

Similarly \( \sqrt{6Y} = \frac{35}{12} \) (same computation, on a different random variable)

So:

\[
6^2(2X - Y) = 4 \cdot \frac{35}{12} + \frac{35}{12} = \frac{175}{12}
\]

Part (b) \( 6^2(X + 3Y - 5) = 6^2X + 3^2 6^2Y + 3 \cdot 6 \cdot 5 = 6^2X + 9 \cdot 6^2Y
\]

\[
= \frac{35}{12} + 108 \frac{35}{12} = \frac{175}{6}
\]