4.75 Light bulbs \( \mu = 900 \) hours
\( \sigma = 50 \) hours

\[ \mu - k\sigma = 700 \]
\[ k = 4 \]

\[ P(700 < X < 1100) \geq 1 - \frac{1}{4^2} \]

Probability random variable \( X \) will assume a value within \( k \) standard deviations of \( \mu \).

\[ P(X \leq 700) \leq (\frac{1}{2})(1 - 0.9375) \leq 0.03125 \]

4.77(b) Random variable \( X \), \( \mu = 10 \), \( \sigma^2 = 4 \) (\( \sigma = 2 \))

\[ P(10 - 10 < X \leq 3) \]
\[ P(-3 < (X - 10) < 3) \]

\[ \frac{P[10 - (\frac{3}{2})^2 < X < 10 + (\frac{3}{2})^2]}{5/9} \]

7.18 \( g(x; p) = p^x \cdot p \cdot \frac{x!}{x!} \)

Moment generating function of \( X \)

\[ M_X(t) = E(e^{tx}) \quad (\S\ 7.3, \text{ Definition 7.2}) \]

\[ = \sum_{x} e^{tx}(p \cdot q^{x-1}) \text{ DISCRETE} \]

\[ = \frac{pe^t}{1-qe^t} \]

\[ \text{MEAN} \]

\[ \mu = \mu_1 \]
\[ \sigma^2 = \mu_2 - \mu^2 \]

\[ \text{VARIANCE} \]

Using Theorem 7.6 in \( \S\ 7.3 \)

\[ \frac{d^r M_X(t)}{dt^r} \bigg|_{t=0} = \mu_r \]

Using sum of infinite geometric series beginning at \( x = 1 \)

\[ \sum_{x=1}^{\infty} r^x = \frac{r}{1-r} \quad |r| < 1 \]
\[ \frac{d}{dt} (\phi(t)) \bigg|_{t=0} = \frac{pe^t}{1-qe^t} + pe^t \left( \frac{qe^t}{1-qe^t} \right)^2 \bigg|_{t=0} \] (using product rule & chain rule)

\[ \mu_1 = \frac{pe^{t}(1-qe^t)}{(1-qe^t)^2} \bigg|_{t=0} = \frac{(1-q)p + pq}{(1-q)^2} = \frac{p}{p+q} \] (since \( p+q = 1 \))

\[ \mu_2 = \frac{d^2(\phi(t))}{dt^2} \bigg|_{t=0} = \frac{(1-qe^t)^2 pe^t + 2pqe^{2t}(1-qe^t)}{(1-qe^t)^4} \bigg|_{t=0} = \frac{2-p}{p^2} \]

\[ \sigma^2 = \mu_2 - \mu_1^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2} \]

7.21 Moment generating function of random variable \( X \) having chi-squared distribution with \( v \) degrees of freedom.

\[ M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx \text{ moment generating function} \]

\[ \chi^2: f(x; \nu) = \begin{cases} \frac{1}{\Gamma(\nu/2) 2^{\nu/2}} \, x^{\nu/2-1} e^{-x/2}, & x > 0 \\ 0, & \text{elsewhere} \end{cases} \]

So for \( \chi^2 \) \( M_X(t) = \frac{1}{\Gamma(\nu/2) 2^{\nu/2}} \int_0^{\infty} e^{tx} x^{\nu/2-1} e^{-x/2} \, dx \)

\[ = \frac{1}{\Gamma(\nu/2) 2^{\nu/2}} \int_0^{\infty} e^{(\nu/2) y} y^{\nu/2-1} e^{-y} 2 \, dy \]

\[ = \left( \frac{2}{1-2t} \right)^{\nu/2} \int_0^{\infty} y^{\nu/2-1} e^{-y} \, dy \]

\[ = \frac{1}{\Gamma(\nu/2) 2^{\nu/2}} \left( \frac{2}{1-2t} \right)^{\nu/2} \Gamma(\nu/2) \]

\[ = (1-2t)^{-\nu/2} \]
7.23 If both $X$ and $Y$ are distributed independently, following exponential distributions with mean parameter $1$, find distribution of

$$U = X + Y$$

$$f(x; 1) = e^{-x}, \quad x > 0$$

→ Using Theorem 7.10

$$M_U(t) = M_X(t)M_Y(t)$$

$$M_X(t) = \mathbb{E}(e^{tx}) = \int_0^\infty e^{tx} e^{-x} \, dx, \quad x > 0$$

Same result for $M_Y(t)$

$$= \int_0^\infty x(t-1) \, dx = \frac{1}{t-1} \left|_0^\infty \right. = \frac{1}{(t-1)^2} = \frac{1}{1-t}$$

$$M_U(t) = \left( \frac{1}{1-t} \right) \left( \frac{1}{1-t} \right) = \frac{1}{(1-t)^2}$$

This is equal to the moment generating function of a gamma distribution with:

$$\alpha = 2, \quad \beta = 1 \quad f(x; \alpha, \beta = 2, \beta = 1)$$

gamma distribution

* The MGF of the gamma distribution is:

$$\left( \frac{1}{1-\beta t} \right)^\alpha$$

Gamma PDF

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}}{\Gamma(\alpha) \beta^\alpha} e^{-\frac{x}{\beta}}$$

$$x > 0, \quad \beta > 0$$

$$M_X(t) = \int_0^\infty e^{tx} f(x) \, dx$$

$$= \frac{1}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-\frac{x}{\beta}} \, dx$$

$$= \frac{1}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty y^{\alpha-1} e^{-y} \left( \frac{\beta}{1-\beta t} \right) \, dy$$

Use a change of variable technique

$$y = x \left( \frac{1}{\beta} - t \right) \quad x = \frac{\beta}{1-\beta t} y \quad dx = \frac{\beta}{1-\beta t} \, dy$$

$$= \frac{1}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty \left( \frac{\beta}{1-\beta t} \right)^\alpha e^{-\frac{\beta y}{1-\beta t}} \left( \frac{1}{1-\beta t} \right) \left( \frac{\beta}{1-\beta t} \right)^\alpha \, dy$$

3/6
8.1  
(a) Population could be all residents of Richmond with phones
(b) Population = infinite number of coin tosses
(c) All pairs of new shoes tested on professional tour.
(d) All time intervals to drive from home to office.

8.14 Sample variance unchanged if constant \( c \) is added to each value

\[ S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \]

Adding constant \( c \) to each \( X_i \): \( \bar{X} + c \)

\[ S^2 = \frac{1}{n-1} \sum_{i=1}^{n} ((X_i + c) - (\bar{X} + c))^2 \]

\[ = \frac{1}{n-1} \sum_{i=1}^{n} c^2 (X_i - \bar{X})^2 \]

\[ = c^2 S^2 \]

8.17 Samples of size \( n = 16 \) drawn from normal population with \( \mu = 50 \), \( \sigma = 5 \)
Probability sample mean \( (\mu_X - 1.9\sigma_X < \bar{X} < \mu_X - 0.4\sigma_X) \)

\[ P(-1.9 < Z < -0.4) = 0.3446 - 0.0287 = 0.3159 \]

8.20 \( f(x) = \begin{cases} \frac{1}{3} & x = 2, 4, 6 \\ 0 & \text{otherwise} \end{cases} \)
Random sample \( n = 54 \)

\[ \mu = E(X) = \sum_x x f(x) = 2 \left(\frac{1}{3}\right) + 4 \left(\frac{1}{3}\right) + 6 \left(\frac{1}{3}\right) = 4 = \mu_X \]

\[ \sigma^2 = E[(X-\mu)^2] = (-2)^2 \left(\frac{1}{3}\right) + (2)^2 \left(\frac{1}{3}\right) = \frac{8}{3} \]

\[ \sigma^2 = \frac{8}{3} \]

\[ \sigma_X = \frac{\sigma}{\sqrt{n}} = \frac{\frac{8}{3}}{54} \]

\[ Z = \frac{4.15 - 4}{0.68} = 0.68 \]

\[ Z = \frac{4.35 - 4}{0.75} = 1.58 \]

\( \Rightarrow P(0.68 < Z < 1.58) \)

\[ = 0.9429 - 0.7517 = 0.1912 \]

Correction for discrete distribution measured to nearest tenth \((0.05)\)
8.32 \( \sigma^2 = 1 \)
\( n = 36 \)

Two different machines
A \( \bar{x}_A = 4.5 \) (ounces)
B \( \bar{x}_B = 4.7 \)

a) \( P(\bar{x}_B - \bar{x}_A \geq 0.2) \) using CLT if \( \mu_A = \mu_B \)

\( \frac{\bar{x}_A - \bar{x}_B}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{n}}} = \frac{-0.2}{\sqrt{\frac{1}{36} + \frac{1}{36}}} \)

Using Theorem 8.3

\[ Z = \frac{(\bar{x}_A - \bar{x}_B) - (\mu_A - \mu_B)}{\sqrt{\frac{\sigma_A^2}{n} + \frac{\sigma_B^2}{n}}} \]

\[ = \frac{-0.2}{\sqrt{\frac{1}{36} + \frac{1}{36}}} \]

\[ = 0.1977 \]

b) Are the population means for the 2 machines actually different?

\[ \rightarrow \] Not necessarily, probability in a) is \textit{not} negligible.

8.33 \( \bar{X} = \overline{X} - \mu \) if \( \overline{X} = \mu \) (population mean) \( n = 25 \)

a) \( \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} = 0.5 \)

b) Observed \( \bar{X} = 7960 \)

\( P(\bar{X} > 7960 | \mu = 7950) \)

\[ Z = \frac{7960 - 7950}{100 / \sqrt{25}} \]

\[ = 0.5 \]

\( P(Z \geq 0.5) = 1 - 0.6915 = 0.3085 \)

not strong evidence that population mean exceeds limit

8.35 \( P(\bar{X} \leq 775 | \mu = 800) \)

\[ Z = \frac{775 - 800}{40 / \sqrt{16}} = -2.5 \]

\( P(Z \leq 2.5) = 0.0062 \)

\( P(\bar{X} \leq 775 | \mu = 760) \)

\[ Z = \frac{775 - 760}{40 / \sqrt{16}} = 1.5 \]

\( P(Z \leq 1.5) = 0.9332 \)

From ex. 8.4 \( n = 16 \)
\( \sigma = 40 \)

The event of observing a sample mean \( \bar{x} \leq 775 \)
is very rare if population mean is 800, making the claim \( \mu = 800 \) questionable.
8.36 \( X_1, X_2, X_n \) \( \sim \) random sample from distribution that can only take on positive values

show \( Y = X_1 X_2 \ldots X_n \) has approx. lognormal distribution

random variable \( X \) has lognormal distribution if \( Y = \ln(X) \) has normal distribution with mean \( \mu \) and s.d. \( \sigma \)

Let \( W_i = \ln(X_i) \)

Using CLT \( Z = \frac{\bar{W} - \mu_W}{\sigma_W / \sqrt{n}} \sim N(0, 1) \)

\( \bar{W} = \frac{1}{n} \sum_{i=1}^{n} \ln(X_i) = \frac{1}{n} \ln(\prod_{i=1}^{n} X_i) = \frac{1}{n} \ln(Y) \)

8.40 \( \chi^2 \) distribution: find \( \chi^2 \) such that

- a) \( P(\chi^2 > \chi^2_{a}) = 0.01 \) when \( \nu = 21 \) \( \chi^2 = 38.932 \)
- b) \( P(\chi^2 < \chi^2_{a}) = 0.95 \) when \( \nu = 6 \) \( \chi^2 = 12.592 \)
- c) \( P(\chi^2_{a} < \chi^2 < 23.209) = 0.015 \) when \( \nu = 10 \)

\[ \chi^2 = 20.483 \]

8.41 Sample variances are continuous measurements

Random sample of 25 observations from normal population \( \sigma^2 = 6 \)

\( a) P\left(S^2 > 9.1\right) = P\left(\frac{(n-1)S^2}{\sigma^2} > \frac{(24)(9.1)}{6}\right) = P(\chi^2 > 36.4) = 0.05 \)

\( b) P\left(3.462 < S^2 < 10.745\right) = P\left(\frac{(24)(3.462)}{6} < \frac{(n-1)S^2}{\sigma^2} < \frac{(24)(10.745)}{6}\right) = P\left(13.848 < \chi^2 < 42.980\right) = 0.95 - 0.01 = 0.94 \)