Solution to the Problem Set 7

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1 Solution to Problem 5.8

From the Fourier series table we have if \( g(t) = \text{rect}(t) \), i.e., a rectangle in time domain which is equal to 1 in \([-0.5, 0.5]\) and zero everywhere else, then \( G(f) = \text{sinc}(f) \).

For this question, \( h(t) \) which is defined in (5.9) of the book, can be assumed to be \( g(t) \) which is first scaled by \( \frac{1}{T} \) in time domain, and then shifted to the right by \( \frac{T}{2} \) seconds. Assuming \( g_T(t) \) as the scaled version of \( g(t) \) by \( \frac{1}{T} \), we simple have \( h(t) = g_T(t - 0.5T) \). It follows that

\[
g_T(t) = g\left(\frac{t}{T}\right) \Rightarrow G_T(f) = Tsinc(fT). \tag{1}
\]

On the other hand, \( H(f) = G_T(f)e^{-j\pi fT} = Tsinc(fT)e^{-j\pi fT} \).

For the second part we have

\[
\lim_{T \to 0} \frac{H(f)}{T} = \lim_{T \to 0} \text{sinc}(fT)e^{-j\pi fT} = 1, \tag{2}
\]

i.e., the Fourier transform tends to that of the unit delta function \( \delta(t) \). Note that the pulse \( h(t) = T \) converges to a delta function as \( T \to 0 \).

2 Solution to Problem 5.12

a) : By definition, the PAM signal is defined as

\[
s_{PAM}(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s) = \left( \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s) \right) * h(t)
= s_\delta(t) * h(t). \tag{3}
\]

where \( s_\delta(t) \) is the signal sampled using a spike train. For \( S_\delta(f) \) we simply have

\[
S_\delta(f) = F_s \sum_{n=-\infty}^{\infty} M(f - nF_s) = \sum_{n=-\infty}^{\infty} \frac{A_m}{2} \left( \delta(f - 0.2 - n) + \delta(f + 0.2 - n) \right). \tag{4}
\]

On the other hand, we have

\[
S_{PAM}(f) = \frac{A_m}{2} \sum_{n=-\infty}^{\infty} \left( H(0.2 + n)\delta(f - 0.2 - n) + H(-0.2 - n)\delta(f + 0.2 - n) \right). \tag{5}
\]

with \( H(f) = Tsinc(fT) \) and \( T = 0.45 \).

b) : Without the equalizer, the ideal low-pass filter passes the frequencies between \(-0.2\) to \(0.2\) Hz. Then the output, namely, \( \hat{m}(t) \), is

\[
\hat{m}(t) = H(0.2)A_m\cos(0.4\pi t) = 0.44A_m\cos(0.4\pi t). \tag{6}
\]

If we use an equalizer, the original signal is obtained, i.e., \( \hat{m}(t) = m(t) \).
3 Solution to Problem 5.13

From page 201, following equation (5.17), the distortion for PAM system can be equalized by assuming the inverse filter for the pulse over the signal bandwidth. We known that (see Problem 5.8), for the rectangular pulse of duration $T$, i.e., $g_T(t)$, the Fourier transform is $G_T(f) = T \text{sinc}(fT)$. Then, at the highest frequency of the signal which is $F_s/2$, we have $G_T(F_s/2) = T \text{sinc}(F_sT/2) = T \text{sinc}(\frac{T}{T_s})$. The ideal low-pass filter has a gain $1/T$ and therefore, the part equalizer has to fix is $\text{sinc}(\frac{T}{T_s})$. For the given value of $T/T_s = 0.25$, the distortion is $\text{sinc}(0.125) = 0.97$ and therefore, the filter gain has to be $H(F_s/2) \approx 1/0.97 \approx 1.03$. The required equalizing gain for different values of $T/T_s$ is plotted in Fig. 1.

4 Solution to Problem 5.14

a) : The Nyquist rate for $s_1(t)$ and $s_2(t)$ is 160 Hz. Therefore, $\frac{2400}{2^m} > 160$ Hz leads to $R \leq 3$.
b) The multiplexing should take place in the following order: every \( \frac{1}{2400} \) s there is a transmission as follows:

- Time 0 to \( \frac{1}{2400} \) s: Transmit \( s_3, s_4, s_1 \).
- Time \( \frac{1}{2400} \) s to \( \frac{1}{2400} \) s: Transmit \( s_3, s_4, s_2 \).
- For the next 6 intervals of length \( \frac{1}{2400} \) s reaching time \( \frac{1}{300} \) s: Transmit \( s_3, s_4, X \), where \( X \) is null.
- Repeat the same procedure every \( \frac{1}{300} \) s.

5 Solution to Problem 5.15

The first quantizer is a midtread quantizer with 7 levels. The second quantizer is a midrise quantizer with 8 levels. Both quantizers require 3 bits for a binary representation. The results from two quantizers are illustrated in Fig. 2.
6 Solution to Problem 5.25

a) : The physical signals are limited in duration, which leads to unlimited bandwidth. Therefore, by assuming a finite sampling frequency, there is always some distortions due to overlapping. However, before sampling the signal, the signal is subject to a low pass filter. Therefore, the distortion caused in the reconstructed signal is due to canceling the high frequency components.

b) : Most signals, such as multimedia signals, can be well approximated by a bandlimited version. Then, by sampling at a high enough rate, the signal can be reconstructed with minimal errors which are not discernable by a human user, e.g., voice signal transmission over a cellular network, PCM, etc.

7 Solution to Problem 5.26

We are multiplying the signal \( g(t) \) by a pulse train \( c(t) \) where the period of \( c(t) \) is \( T_s = \frac{1}{f_s} \), and the duration of pulse is \( T \). Naturally, \( T < \frac{1}{f_s} \), so \( f_s T < 1 \) (I think there is a typo in the book). So we proceed with the assumption that \( f_s T \ll 1 \).

a) : You can represent \( c(t) \) as \( c(t) = p(t) \ast d(t) \) where \( p(t) \) is a single pulse with duration \( T \) and \( d(t) \) is a delta train at intervals \( T_s \), i.e., \( d(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \). Then, by the properties of Fourier transform \( C(f) = P(f)D(f) \). We know that (problem 5.8) \( P(f) = T \text{sinc}(fT) \). On the other hand, \( D(f) = f_s \sum_{k=-\infty}^{\infty} \delta(f - kT_s) \). Then, we find that \( C(f) = \sum_{k=-\infty}^{\infty} a_k \delta(f - kT_s) \) with \( a_k = f_s T \text{sinc}(kf_s T) \).

On the other hand, we are interested in the spectrum of \( g_c(t) = g(t) c(t) \). Again, from the Fourier transform, we have \( G_c(f) = G(f) \ast C(f) \). Given that \( C(f) \) is a series of delta functions in the frequency domain, the result of the convolution is immediate. We obtain \( G_c(f) = \sum_{k=-\infty}^{\infty} a_k G_c(f - kT_s) \).

b) : Naturally, if \( f_s \) satisfies the Nyquist theorem, and \( a_0 \neq 0 \), the signal can be recovered by using the ideal low-pass filter with a bandwidth \( f_s/2 \). Note that the pulse train \( c(t) \) has a nonzero DC value with \( a_0 = T/T_s \), i.e., the duty cycle. The scenario with a zero DC value for \( c(t) \) occurs in the next problem.

8 Solution to Problem 5.27

The output input relation between \( y(t) \) and \( x(t) \) can be expressed as \( y(t) = x(t) \times c(t) \) where \( c(t) \) is a periodic square waveform, where \( c(t) = 1, t \in \lbrack 2kT_s, 2kT_s + T_s \rbrack \), \( c(t) = -1, t \in \lbrack 2kT_s + T_s, 2kT_s + 2T_s \rbrack \). First note that this is some form of modulation. The easiest way to get \( x(t) \) from \( y(t) \) is to multiply \( y(t) \) with the same modulating waveform \( c(t) \), i.e., \( y(t)c(t) = x(t)c(t)^2 = x(t) \).
To find the spectrum of $Y(f)$ note that $Y(f) = X(f) * C(f)$. For $C(f)$ we have

$$c(t) = \sum_{k=-\infty}^{\infty} a_k e^{\frac{jk \pi t}{T_s}} \Rightarrow C(f) = \sum_{k=-\infty}^{\infty} a_k \delta(f - \frac{2k}{T_s}). \quad (7)$$

For the given square wave, it turns out that $a_{2k} = 0$ and $a_{2k+1}$ is imaginary. Then, we have

$$Y(f) = \sum_{k=-\infty}^{\infty} a_{2k+1} X(f - \frac{k}{T_s}). \quad (8)$$

9 Solution to Problem 5.28

The input signal is bandlimited and has sinusoid tones at frequencies, $f = kf_0$ where $k = 1, 2, ..., m$. Assume $X(f) = \sum_{k=-m}^{m} a_k \delta(f - kf_0)$.

a) : Consider $f_s = (1 - a)f_0$ and assume we have a single-tone $z(t)$ such that $Z(f) = \delta(f - kf_0)$. After sampling $z(t)$ we have

$$Z_d(f) = \sum_{n=-\infty}^{\infty} \delta(f - kf_0 - nf_s). \quad (9)$$

We have $kf_0 + nf_s = (k + n)f_0 - nf_0$. Then, for $n = -k$ we will have a spike at $f = -naf_0 = kf_0$. Similarly, if we assume $Z(f) = \delta(f + kf_0)$, then the same argument says that there is a spike at $f = -kf_0$. Since we assumed an arbitrary $k$, this is true for all $k = 1, 2, 3, ...$. Therefore, for very small $a$, we have spikes at $akf_0$.

Now if we filter-out only these spikes, and call the signal $y(t)$, we have $Y(f) = \sum_k a_k \delta(f - kf_0)$. If we normalize $Y(f)$ by $\frac{1}{2}$, then we can assume that $Y(f)$ is the output to the system $y(t) = x(at)$ (it follows from the properties of the Fourier transform).

b) : The highest frequency of $Y(f)$ is at $mf_0$. We can obtain $Y(f)$ by filtering $X(f)$ after sampling using an ideal filter with a bandwidth of $f_s/2$. Then, the highest frequency component of $Y(f)$ must be smaller than filter bandwidth, i.e.,

$$mf_0 \leq \frac{f_s}{2} = \frac{(1 - a)f_0}{2} \Rightarrow a \leq \frac{1}{2m + 1}. \quad (10)$$

c) : This follows from the argument of part a. Note that when a signal is expanded in time, then it is compressed in frequency.

10 Solution to Problem 5.30

a) : You can find the transform by direct integration of the given waveform. I explain an alternative method (which is much faster, although it seems otherwise). First reconstruct the signal as follows:
• Take a rectangular pulse of duration $T$, say, $p(t)$. Define the rectangular pulse
  $A(t) = p(t) \ast p(t)$.

• Note that in time domain, we have $A(t) = (t + T)u(t + T) - 2tu(t) + (t - T)u(t - T)$.

• Add the signal $A(t)$ with $T$ times its derivative. Note that $A'(t) = u(t + T) - 2u(t) + u(t - T)$.

• Shift the result, in time, to the right, by $T$.

• Divide the result of previous steps by $T$.

• You find the signal in Fig. 5.29 of the book. Then, we have
  $h(t) = \frac{1}{T} (A(t - T) + TA'(t - T))$.

Using this method of constructing $h(t)$ you can easily find the Fourier transform. We simply have

\[
H(f) = \frac{1}{T} \left( A(f)e^{-j2\pi fT} + j2\pi fTA(f)e^{-j2\pi fT} \right)
= \frac{A(f)}{T} (1 + j2\pi fT) e^{-j2\pi fT}
= \frac{(P(f))^2}{T} (1 + j2\pi fT) e^{-j2\pi fT}
= T^{-1} (T \text{sinc}(fT))^2 (1 + j2\pi fT)
= T^{-1} \left( \frac{1 - \exp(-j2\pi fT)}{j2\pi fT} \right)^2 (1 + j2\pi fT).
\]

Note that in the last step, I simply used the fact that by shifting the pulse from $-T/2$ to $T/2$, by $T/2$ to the right, you get a pulse starting from 0 and ending at $T$. For such a pulse, the Fourier transform is $\frac{1 - \exp(-j2\pi fT)}{j2\pi fT}$.

b) : The magnitude response is sketched in Fig. 3 and the phase response in Fig. 4. The plots are for the case $T = 1$.

c) : At the frequency $f = \frac{f_s}{2} = \frac{1}{2T_s}$, we need to compensate for $H(\frac{f}{2T_s})$. For $T/T_s = 0.1$ we have $|H(0.05)| \approx 1.04$ which is roughly the same as the zero-order hold (Actually worse). There is also a phase compensation requirement as well. As you can see from Fig. 4, the phase around origin is nonlinear!

Remark: If you look up the first-order hold filters in the literature, specifically, in the digital control systems literature where these types of sampling and reconstruction are crucial, you realize that the first-order-hold is defined differently! In fact, the first-order hold filter remembers the $A(t)$ explained in part a.
Figure 3: Magnitude response.

Figure 4: Phase response.
d) I assumed three different types of filters: 1) zero-order hold, 2) first-order hold as defined here, and 3) the actual first-order hold in my opinion. Fig. ?? shows the signal along with its samples only. Fig. 7, you see the signal, the samples, the output of zero-order hold, and the first-order hold according to the definition of this problem and in the following plot, the first-order hold system used in control systems literature. As anticipated, the actual first-order hold system should outperform the zero-order hold system.
Figure 6: The output of reconstruction filter for zero and first-order hold systems (according to the book).

Figure 7: The output of reconstruction filter for zero and first-order hold systems (according to the literature).