Properties of the Fourier Transform

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Reference:

Sections 2.2 - 2.3 of

S. Haykin and M. Moher, Introduction to Analog & Digital Communications, 2nd ed., John Wiley & Sons, Inc., 2007. ISBN-13 978-0-471-43222-7.

The Fourier Transform (FT)

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt$$
$$g(t) = \int_{-\infty}^{\infty} G(f) e^{+j2\pi ft} df$$

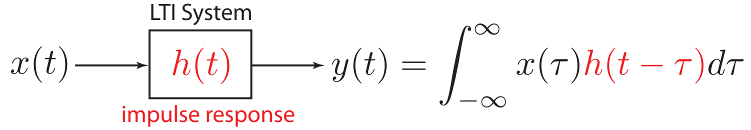
Notation:

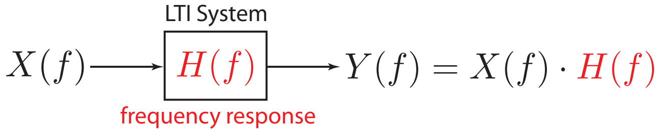
$$g(t) \rightleftharpoons G(f)$$
$$G(f) = \mathbf{F}[g(t)]$$
$$g(t) = \mathbf{F}^{-1}[G(f)]$$

Importance of FT Theorems and Properties

- We *live* in the time-domain.
- However, sometimes viewing information signals or system operation as function of time does not easily provide insight.
- The Fourier transform converts a signal or system representation to the frequency-domain, which provides another way to *visualize* a signal or system convenient for analysis and design.
- The properties of the Fourier transform provide valuable insight into how signal operations in the time-domain are described in the frequency-domain.

Importance of FT Theorems and Properties





- For systems that are <u>linear time-invariant</u> (LTI), the Fourier transform provides a decoupled description of the system operation on the input signal much like when we diagonalize a matrix.

FT Theorems and Properties

Property/Theorem	Time Domain		Frequency Domain
Notation:	$g(t)$	$\rightleftharpoons$	$G(f)$
	$g_1(t)$	$\rightleftharpoons$	$G_1(f)$
	$g_2(t)$	$\rightleftharpoons$	$G_2(f)$
Linearity:	$c_1 g_1(t) + c_2 g_2(t)$	$\rightleftharpoons$	$c_1 G_1(f) + c_2 G_2(f)$
Dilation:	$g(at)$	$\rightleftharpoons$	$\frac{1}{\|a\|} G\left(\frac{f}{a}\right)$
Conjugation:	$g^*(t)$	$\rightleftharpoons$	$G^*(-f)$
Duality:	$G(t)$	$\rightleftharpoons$	$g(-f)$
Time Shifting:	$g(t-t_0)$	$\rightleftharpoons$	$G(f)e^{-j2\pi f t_0}$
Frequency Shifting:	$e^{j2\pi f_c t} g(t)$	$\rightleftharpoons$	$G(f-f_c)$
Area Under $G(f)$:	$g(0)$	$=$	$\int_{-\infty}^{\infty} G(f)df$
Area Under $g(t)$:	$\int_{-\infty}^{\infty} g(t)dt$	$=$	$G(0)$
Time Differentiation:	$\frac{d}{dt}g(t)$	$\rightleftharpoons$	$j2\pi f G(f)$
Time Integration :	$\int_{-\infty}^{t} g(\tau)d\tau$	$\rightleftharpoons$	$\frac{1}{j2\pi f} G(f)$
Modulation Theorem:	$g_1(t)g_2(t)$	$\rightleftharpoons$	$\int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda$
Convolution Theorem:	$\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau$	$\rightleftharpoons$	$G_1(f)G_2(f)$
Correlation Theorem:	$\int_{-\infty}^{\infty} g_1(t)g_2^*(t-\tau)dt$	$\rightleftharpoons$	$G_1(f)G_2^*(f)$
Rayleigh's Energy Theorem:	$\int_{\infty}^{\infty} \|g(t)\|^2 dt$	$=$	$\int_{\infty}^{\infty} \|G(f)\|^2 df$

Dilation Property

$$\boxed{g(at) \rightleftharpoons \frac{1}{|a|} G\left(\frac{f}{a}\right)}$$

Proof: Let $h(t) = g(at)$ and $H(f) = \mathbf{F}[h(t)]$.

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} g(at)e^{-j2\pi ft}dt \end{aligned}$$

Idea: Do a change of integrating variable to make it look more like $G(f)$.

Dilation Property

Let $\tau = at$. Assume for now $a > 0$ and finite.
Three things must be changed:

1. integrand: substitute $t = \tau/a$.
2. limits: for $t = \infty$, $\tau = \infty$; for $t = -\infty$, $\tau = -\infty$.
3. differential: $d\tau = a dt$ or $dt = d\tau/a$.

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} g(at)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} g(\tau)e^{-j2\pi f\tau/a}\frac{d\tau}{a} \\ &= \frac{1}{a}\int_{-\infty}^{\infty} g(t)e^{-j2\pi\left(\frac{f}{a}\right)t}dt = \frac{1}{a}G\left(\frac{f}{a}\right) \end{aligned}$$

Dilation Property

For $a < 0$ and finite, all remains the same except the integration limits:

1. integrand: substitute $t = \tau/a$.
2. limits: for $t = \infty$, $\tau = -\infty$; for $t = -\infty$, $\tau = +\infty$.
3. differential: $d\tau = a dt$ or $dt = d\tau/a$.

Therefore,

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} g(at) e^{-j2\pi f t} dt = \int_{+\infty}^{-\infty} g(\tau) e^{-j2\pi f \tau/a} \frac{d\tau}{a} \\ &= -\frac{1}{a} \int_{-\infty}^{\infty} g(t) e^{-j2\pi \left(\frac{f}{a}\right) t} dt = -\frac{1}{a} G\left(\frac{f}{a}\right) \end{aligned}$$

Dilation Property

Therefore,

$$H(f) = \begin{cases} +\frac{1}{a} G\left(\frac{f}{a}\right) & a > 0 \\ -\frac{1}{a} G\left(\frac{f}{a}\right) & a < 0 \end{cases} = \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

for $a \neq 0$, and

$$h(t) = g(at) \quad \rightleftharpoons \quad H(f) = \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

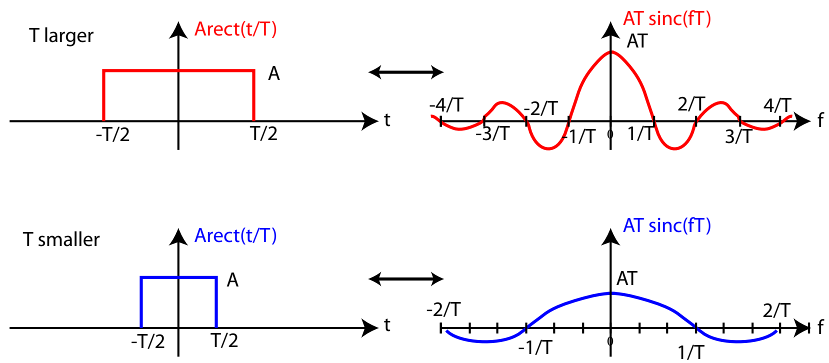
Inverse Relationship

$$\boxed{g(at) \rightleftharpoons \frac{1}{|a|} G\left(\frac{f}{a}\right)}$$

- A stretch in the time (or frequency) domain by a given factor a leads to a compression in the frequency (or time) domain by same same factor a.
- There is also a corresponding amplitude change in the frequency domain.
  - This is needed to keep the energies of the signals in both domains equated (from Rayleigh's Energy Theorem):

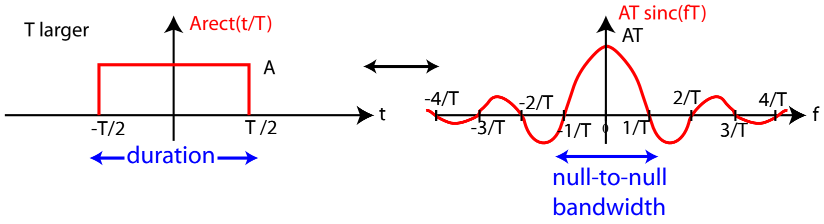
$$\int_{\infty}^{\infty} |g(t)|^2 dt = \int_{\infty}^{\infty} |G(f)|^2 df$$

Inverse Relationship



Time-Bandwidth Product

time-duration of a signal $\times$ frequency bandwidth $=$ constant



Time-Bandwidth Product

time-duration of a signal $\times$ frequency bandwidth $=$ constant

- manifestation of the inverse relationship
- the constant depends on the definitions of duration and bandwidth and can change with the shape of signals being considered
- It can be shown that: time-duration of a signal $\times$ frequency bandwidth $\geq \frac{1}{4\pi}$ with equality achieved for a Gaussian pulse.

Time Shifting Property

$$g(t - t_0) \rightleftharpoons G(f)e^{-j2\pi f t_0}$$

Proof: Let $h(t) = g(t - t_0)$ and $H(f) = \mathbf{F}[h(t)]$.

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi ft}dt$$

Idea: Do a change of integrating variable to make it look more like $G(f)$.

Time Shifting Property

Let $\tau = t - t_0$ where $t_0 \in \mathbb{R}$. Three things must be changed:

1. integrand: substitute $t = \tau + t_0$.
2. limits: for $t = \infty$, $\tau = \infty$; for $t = -\infty$, $\tau = -\infty$.
3. differential: $d\tau = dt$.

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} g(\tau)e^{-j2\pi f(\tau + t_0)}d\tau \\ &= \int_{-\infty}^{\infty} g(\tau)e^{-j2\pi f\tau} \cdot e^{-j2\pi f t_0}d\tau \\ &= e^{-j2\pi f t_0}\int_{-\infty}^{\infty} g(\tau)e^{-j2\pi f\tau}d\tau = e^{-j2\pi f t_0}G(f) \end{aligned}$$

$$\therefore \quad g(t - t_0) \quad \rightleftharpoons \quad G(f)e^{-j2\pi f t_0}$$

Time Shifting Property

- $G(f)e^{-j2\pi f t_0}$ results in a change of phase only of $G(f)$.

Magnitude:

$$|G(f)e^{-j2\pi f t_0}| = |G(f)|\underbrace{|e^{-j2\pi f t_0}|}_{=1} = \underbrace{|G(f)|}_{\text{no mag change}}$$

Phase:

$$\begin{aligned}\angle[G(f)e^{-j2\pi f t_0}] &= \angle[|G(f)|e^{j\angle G(f)}e^{-j2\pi f t_0}] \\ &= \underbrace{\angle|G(f)|}_{=0} + \angle G(f) + \angle e^{-j2\pi f t_0} \\ &= \underbrace{\angle G(f) - 2\pi f t_0}_{\text{phase change!}}\end{aligned}$$

Time Shifting Property

- Recall, that the phase of the FT determines how the complex sinusoid $e^{j2\pi f t}$ lines up in the synthesis of $g(t)$.
- A delayed signal $g(t - t_0)$, requires all the corresponding sinusoidal components $\{e^{j2\pi f t}\}$ for $-\infty < f < \infty$ to be delayed by t_0 thus changing their individual absolute phases.

Convolution Theorem

$$\boxed{\int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \rightleftharpoons G_1(f)G_2(f)}$$

Proof: Let $H(f) = G_1(f)G_2(f)$.
From the synthesis equation:

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi f t}df = \int_{-\infty}^{\infty} G_1(f)G_2(f)e^{j2\pi f t}df$$

From the analysis equation, substitute in:

$$G_2(f) = \int_{-\infty}^{\infty} g_2(t')e^{-j2\pi f t'}dt'$$

Convolution Theorem

$$\begin{aligned}h(t) &= \int_{-\infty}^{\infty} G_1(f)G_2(f)e^{j2\pi f t}df \\ &= \int_{-\infty}^{\infty} G_1(f)\left[\int_{-\infty}^{\infty} g_2(t')e^{-j2\pi f t'}dt'\right]e^{j2\pi f t}df \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(f)g_2(t')e^{j2\pi f(t-t')}dt'df\end{aligned}$$

Idea: Substitute for the integrating variable t'.

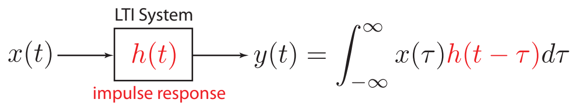
Convolution Theorem

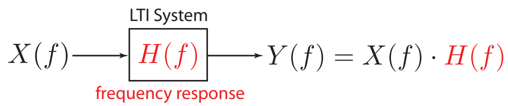
Let $\tau = t - t'$.

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty}\int_{\infty}^{-\infty} G_1(f)g_2(t-\tau)e^{j2\pi f\tau}(-d\tau)df \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(f)g_2(t-\tau)e^{j2\pi f\tau}d\tau df \\ &= \int_{-\infty}^{\infty} g_2(t-\tau)\left[\int_{-\infty}^{\infty} G_1(f)e^{j2\pi f\tau}df\right]d\tau \\ &= \int_{-\infty}^{\infty} g_2(t-\tau)g_1(\tau)d\tau = \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} g_1(\tau)g_2(t-\tau)d\tau \rightleftharpoons G_1(f)G_2(f)$$

LTI Systems and Filtering





- The convolution theorem provides a filtering perspective to how a linear time-invariant system operates on an input signal.
- The LTI system scales the sinusoidal component corresponding to frequency f by $H(f)$ providing frequency selectivity.

Conjugation Property and Conjugate Symmetry

$$\boxed{g^*(t) \rightleftharpoons G^*(-f)}$$

If $g(t)$ is real (i.e., not complex), then we can say:

$$\begin{aligned} g(t) &= g^*(t) \\ \mathbf{F}[g(t)] &= \mathbf{F}[g^*(t)] \\ G(f) &= G^*(-f) \end{aligned}$$

That is, $G(f)$ obeys conjugate symmetry.

Conjugate Symmetry

$$\begin{aligned} G(f) &= G^*(-f) \\ |G(f)| &= |G^*(-f)| = |G(-f)| \\ \angle G(f) &= \angle G^*(-f) = -\angle G(-f) \end{aligned}$$

Therefore,

$$\begin{aligned} |G(f)| &= |G(-f)| && \text{mag is EVEN} \\ \angle G(f) &= -\angle G(-f) && \text{phase is ODD} \end{aligned}$$

for real time-domain signals. ■