Chapter 5
Pulse Modulation:
Transition from Analog to Digital Communications

Section 5.1
Sampling

Reference:
Sections 5.1- 5.6 of

From duality:

periodic in time ⇔ discrete in frequency
discrete in time ⇔ periodic in frequency
Sampling

We can model sampling as multiplication of an analog waveform $g(t)$ with an impulse train:

- analog waveform: $g(t)$
- impulse train:
  \[
  \sum_{n=-\infty}^{\infty} \delta(t - nT_S)
  \]
- model of sampling:
  \[
  g(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_S) = \sum_{n=-\infty}^{\infty} g(nT_S) \delta(t - nT_S)
  \]

Note: $g(t)$ contains the information of $g(nT_S)$ and represents a good model of sampling.

\[
\begin{align*}
g(t) & \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(f) \\
g(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_S) & \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(f) \ast \frac{1}{T_S} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T_S}) \\
g(t) & \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(f) = \frac{1}{T_S} \sum_{k=-\infty}^{\infty} G(f) \ast \delta(f - \frac{k}{T_S}) \\
g(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_S) & \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(f) = \frac{1}{T_S} \sum_{k=-\infty}^{\infty} G(f) \ast \delta(f - \frac{k}{T_S}) \\
g(t) & \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad G(f) = \frac{1}{T_S} \sum_{k=-\infty}^{\infty} G(f - \frac{k}{T_S})
\end{align*}
\]
Let $f_s = \frac{1}{T_S}$:

$$g_0(t) = \sum_{n=-\infty}^{\infty} g(nT_S) \delta(t - nT_S) \iff G_0(f) = \frac{1}{T_S} \sum_{k=-\infty}^{\infty} G(f - \frac{k}{T_S})$$

$$g_0(t) = \sum_{n=-\infty}^{\infty} g(nT_S) \delta(t - nT_S) \iff G_0(f) = f_s \sum_{k=-\infty}^{\infty} G(f - kf_S)$$

Recall, $\delta(t - t_0) \iff e^{j2\pi ft_0}$

$$G_0(f) = \mathcal{F}[g_0(t)] = \mathcal{F} \left[ \sum_{n=-\infty}^{\infty} g(nT_S) \delta(t - nT_S) \right]$$

$$= \sum_{n=-\infty}^{\infty} g(nT_S) \mathcal{F}[\delta(t - nT_S)] = \sum_{n=-\infty}^{\infty} g(nT_S) e^{j2\pi nT_Sf}$$

Therefore,

$$G_0(f) = \sum_{n=-\infty}^{\infty} g(nT_S) e^{j2\pi nT_Sf} = f_s \sum_{k=-\infty}^{\infty} G(f - kf_S)$$
Section 5.1

ALIASING

Increasing $T_S$

$A/T$ $1/T-1/T$ $0$

$A/T$ $1/T$ $2/T-1/T-2/T$ $0$


ALIASING

Increasing $T_S$

$S$

$S$

$S$

$A/T S$

$2A/T S$

$S$

$S SS S$

$S SS SS SS S$

$|G(f)|$

$|G(f)|$

$|G(f)|$

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Section 5.1

ANTI-ALIASING

FILTER APPLIED

Increasing $T_S$

$A/T_s$ $1/T-1/T$ $0$

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ANTI-ALIASING

FILTER APPLIED

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ANTI-ALIASING

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Section 5.1

Sampling Theorem

\[ W < \frac{1}{T_S} - W \]
\[ W < f_S - W \]
\[ f_S > 2W = \text{Nyquist Rate} \]

OR

\[ f_S > 2W = \text{Nyquist Rate} \]
\[ T_S < \frac{1}{2W} = \text{Maximum Sampling Period} \]
Sampling Theorem

Suppose that a signal $g(t)$ is strictly band-limited with no frequency components higher than $W$ Hz. That is, $G(f)$ is zero for $|f| \geq W$.

Then $g(t)$ can be exactly recovered from its sample values $g(nT_s)$ for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ through band-limited interpolation if:

$$f_s = \frac{1}{T_s} > 2W$$

where $2W$ is called the Nyquist Rate.

We will assume that all message signals $m(t)$ from now on are sampled above the Nyquist Rate.

Pulse Modulation

- the variation of a regularly spaced constant amplitude pulse stream to superimpose information contained in a message signal.

- Three types:
  1. pulse amplitude modulation (PAM)
  2. pulse duration modulation (PDM)
  3. pulse position modulation (PPM)

Pulse Amplitude Modulation (PAM)

Note: $T < T_s$
Pulse Amplitude Modulation

Two steps: Sampling and Hold

1. Instantaneous sampling: the message $m(t)$ is sampled every $T_s$ seconds where $f_s = \frac{1}{T_s}$ obeys the sampling theorem.

2. Lengthening: extending the duration of each sample so that it occupies $T$ seconds.

Sample-and-Hold Analysis

Suppose

$$h(t) = \text{rect} \left( \frac{t - \frac{T}{2}}{T} \right) = \begin{cases} 1 & 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$m_s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s)$$

$$m_s(t) \ast h(t) = \left[ \sum_{n=-\infty}^{\infty} m(nT_s)\delta(t - nT_s) \right] \ast h(t)$$

$$= \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s)$$

Sample-and-Hold Analysis

$$s(t) = m_s(t) \ast h(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s)$$

$$s(t) = m_i(t) \ast h(t)$$

$$S(f) = M_s(f) \cdot H(f)$$

$$h(t) \leftrightarrow F H(f)$$

rectangle $\leftrightarrow F$ sinc
Pulse Amplitude Modulation

Recovery:
- Pass the samples \( s(t) \) through a lowpass filter.

There is a trade-off to the pulse width \( T \):
- The signal lengthening stage reduces the bandwidth of the overall pulse making it more efficient for communications.
- However, this is some distortion when recovering the signal as the sinc function in the frequency domain warps the frequency domain of the information signal.
Pulse Duration Modulation (PDM)

- The width of the pulse reflects the sampled signal amplitude.
- The position of the leading edge, trailing edge or both may be modified to reflect the changing duration of the pulse.
- Also known as: pulse width modulation or pulse length modulation.
- PDM is wasteful of energy when the pulses are long, but the information is only in the pulse transitions.

Pulse Position Modulation

- The position of the pulse reflects the sampled signal amplitude.
- PPM can be represented as:
  \[ s(t) = \sum_{n=-\infty}^{\infty} g(t - nT_s - k_p m(nT_s)) \]
  where
  - \( k_p \) is the sensitivity factor
  - The adjacent pulses must be strictly non-overlapping.
Pulse Position Modulation

\[ s(t) = \sum_{n=-\infty}^{\infty} g(t - nT_s - k_p m(nT_s)) \]

where

\[ k_p |m(t)|_{\max} < \left( \frac{T_s}{2} \right) \]

for strictly non-overlapping pulses.

Pulse Position Modulation

Recovery:

1. Determine pulse centers to decode values of \( \{ m(nT_s) \} \).
2. Use bandlimited interpolation to obtain \( \hat{m}(t) \).

Assuming that the samples \( \{ m(nT_s) \} \) obey the Sampling Theorem, then \( \hat{m}(t) = m(t) \) leading to ideal communication reconstruction.

Analog vs. Digital Communications

- Channel noise and signal distortion on analog communication system is cumulative.
- Regenerative repeaters in digital communication system can practical eliminate degrading effects of channel noise and signal distortion.
- Coding can be used in digital communication systems for greater reliability and security.
- Digital communications requires that we not just sample in time, but quantize in amplitude.
Amplitude Quantization

Amplitude quantization: the process of transforming the sample amplitude $m(nT_s)$ of a baseband signal $m(t)$ at time $t = nT_s$ into a discrete amplitude $v(nT_s)$ taken from a finite set of possible levels.

- non-reversible process

Let us denote $m(nT_s)$ simply as $m$ and $v(nT_s)$ simply as $v$.

Let:

- $m = \text{original discrete-time signal sample}$
- $v = \text{quantized digital signal sample}$
- $g(\cdot) = \text{quantization operator}$
- $e_m = \text{quantization error}$

$v = g(m)$

$e_m = v - m$

Uniform Quantization Example

... result of rounding to the nearest quantization level.
Uniform Quantization: Midtread

Uniform Quantization: Midrise

Uniform Quantization: Midtread vs. Midrise

Pulse-Code Modulation

- Most basic form of digital pulse modulation
- Elements of pulse-code modulation (PCM):
  1. Transmitter
  2. Transmission Path
  3. Receiver

Pulse-Code Modulation Diagram:

SOURCE  \(\rightarrow\) Transmitter  \(\rightarrow\) Transmission Path  \(\rightarrow\) Receiver  \(\rightarrow\) DESTINATION
PCM Transmitter

Continuous-time Message Signal | Anti-aliased Cts-time Signal | Discrete-time Signal | Digital signal | PCM Data Sequence

Source | Low-pass Filter | Sampler | Quantizer | Encoder

Anti-aliasing Filter | Sampling above Nyquist with Narrow Rectangular PAM Pulses | Using a Non-uniform Quantizer | Maps Numbers to Bit Sequences

PCM Transmitter: Sampler

Continuous-time Message Signal | Anti-aliased Cts-time Signal | Discrete-time Signal | Digital signal | PCM Data Sequence

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PCM Transmitter: Non-Uniform Quantizer

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PCM Transmitter: Non-Uniform Quantizer

Compressor: $\mu$-law:

\[ m \equiv \text{message sample} \]
\[ v \equiv \text{quantized value} \]
\[ \log \equiv \text{natural logarithm} \]
\[ |v| = \frac{\log(1 + \mu|m|)}{\log(1 + \mu)} \]

Note: $\mu = 0$ corresponds to a linear quantizer. Typically $\mu \approx 255$ used in practice.
PCM Transmitter: Non-Uniform Quantizer

Compressor: \( \mu \)-law:

\[
|v| = \begin{cases} 
\frac{A|m|}{1 + \log(A)} & 0 \leq |m| \leq \frac{1}{A} \\
\frac{1 + \log(|m|)}{1 + \log(A)} & \frac{1}{A} \leq |m| \leq 1 
\end{cases}
\]

Note: \( A = 1 \) corresponds to a linear quantizer. Typically \( A \approx 100 \) used in practice.

PCM Transmitter: Encoder

- maps quantization-level output to a code word
- typically binary code words are employed
Encoder: Example

8 Quantization-levels, or $R = 3$-bit code words:

<table>
<thead>
<tr>
<th>Quantization-Level Index</th>
<th>Binary Codeword</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>000</td>
</tr>
<tr>
<td>1</td>
<td>001</td>
</tr>
<tr>
<td>2</td>
<td>010</td>
</tr>
<tr>
<td>3</td>
<td>011</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
</tr>
</tbody>
</table>

PCM: Regenerative Repeater

- Amplifier-Equalizer: shapes the received pulses to compensate for amplitude and phase distortions produced by transmission line
- Timing Circuit: produces periodic pulse train derived from received pulses to sample the pulses where signal-to-noise ratio is highest
- Decision-making Device: sample of pulse is compared to a pre-determined threshold
  - if threshold exceed a clean new pulse representing 1 is transmitted
  - otherwise a clean new pulse representing 0 is transmitted
PCM: Receiver

Two Stages:

1. **Decoding and Expanding:**
   1.1 regenerate the pulse one last time and interpret bit sequence
   1.2 group bits into code words
   1.3 interpret code words as quantization level
   1.4 pass level through expander (opposite of compressor)

2. **Reconstruction:**
   2.1 pass expander output through low-pass reconstruction filter (cutoff is equal to message bandwidth) to estimate original message $m(t)$