3.7. Let $y(n) = x_1(n) * x_2(n)$.

Then $Y(Z) = X_1(Z)X_2(Z) \Leftarrow$ "Convolution" Property

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x_1(Z) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\tfrac{1}{3}\right)^n z^{-n} + \sum_{n=-\infty}^{-1} \left(\tfrac{1}{2}\right)^{-n} z^{-n} = \frac{1}{1-\frac{1}{3}Z^{-1}} + \frac{\frac{1}{2}Z}{1-\frac{1}{2}Z}$$

$$= \frac{\frac{5}{6}}{(1-\frac{1}{3}Z^{-1})(1-\frac{1}{2}Z)} \qquad ROC_1: \tfrac{1}{3} < |Z| < 2$$

$$X_2(Z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n} = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n Z^{-n} = \frac{1}{1-\frac{1}{2}Z^{-1}}, \quad ROC_2: |Z| > \tfrac{1}{2}$$

Thus: $Y(Z) = X_1(Z)X_2(Z)$

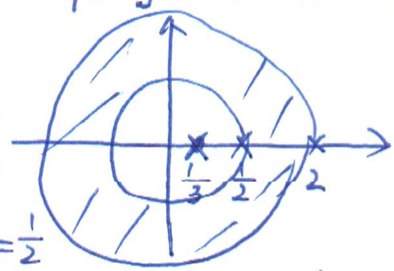
$$= \frac{\frac{5}{6}}{(1-\frac{1}{3}Z^{-1})(1-\frac{1}{2}Z)(1-\frac{1}{2}Z^{-1})}$$

$$= \frac{-\frac{5}{6}(-2Z^{-1})}{(1-\frac{1}{3}Z^{-1})(1-2Z^{-1})(1-\frac{1}{2}Z^{-1})}$$

$$= \frac{-2}{1-\frac{1}{3}Z^{-1}} + \frac{\frac{10}{3}}{1-\frac{1}{2}Z^{-1}} + \frac{-\frac{4}{3}}{1-2Z^{-1}} \Leftarrow \text{Partial Fraction Expansion}$$

$ROC = ROC_1 \cap ROC_2 \Leftarrow$ Property "Convolution"

Thus: $ROC: \frac{1}{2} < |Z| < 2$



Thus $y(n)$ is two-sided,

and the poles $p_1 = \frac{1}{3}$, $p_2 = \frac{1}{2}$ provide the causal parts, $p_3 = 2$ provides the anti-causal part.

Based on $\frac{1}{1-az^{-1}} \xrightarrow[\text{causal}]{|z|>|a|} a^n u(n)$, $\frac{1}{1-az^{-1}} \xrightarrow[\text{anti-causal}]{|z|<|a|} -a^n u(-n-1)$

we have $y(n) = -2\left(\frac{1}{3}\right)^n u(n) + \frac{10}{3}\left(\frac{1}{2}\right)^n u(n) + \frac{4}{3}(2)^n u(-n-1)$.

3.9.

Scaling Theorem: $a^n x(n) \xrightarrow{Z} X(a^{-1}z)$

Thus let $y(n) = e^{j\omega_0 n} x(n) = (e^{j\omega_0})^n x(n)$.

we have $Y(z) = X(e^{-j\omega_0} z)$

Thus poles and zeros are phase rotated by an angle ω_0.

3.12. Note: there is a typo in textbook.

$$X(Z) = \frac{1}{(1-2Z^{-1})(1-Z^{-1})^2}$$

$$\frac{X(Z)}{Z} = \frac{Z^2}{(Z-2)(Z-1)^2} = \frac{A_1}{(Z-2)} + \frac{A_2}{(Z-1)} + \frac{A_3}{(Z-1)^2}$$

$$A_1 = \frac{Z^2}{(Z-1)^2}\Big|_{Z=2} = 4$$

$$A_3 = \frac{Z^2}{(Z-2)}\Big|_{Z=1} = -1$$

$$A_2 = \frac{d}{dz}\left[\frac{Z^2}{(Z-2)}\right]\Big|_{Z=1} = -3$$

$$X(Z) = \frac{4}{1-2Z^{-1}} - \frac{3}{1-Z^{-1}} - \frac{Z^{-1}}{(1-Z^{-1})^2}$$

Based on $a^n u(n) \overset{Z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$ and $n a^n u(n) \overset{Z}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}$

we have $x(n) = [4(2)^n - 3 - n]\, u(n)$.

3.13.

(a) $x_1(n) = \begin{cases} x(\frac{n}{2}) & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} = \sum_{n=\infty}^{\infty} x(\tfrac{n}{2}) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-2k} \Leftarrow k = \tfrac{n}{2}$$

$$= \sum_{k=-\infty}^{\infty} x(k) (z^2)^{-k}$$

$$= X(z^2)$$

(b) $x_2(n) = x(2n)$

$$X_2(n) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(2n) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-\frac{k}{2}} \Leftarrow k=2n., \text{ thus } k \text{ is even}$$

$$= \sum_{k=-\infty}^{\infty} \left[\frac{x(k) + (-1)^k x(k)}{2} \right] z^{-\frac{k}{2}}, \Leftarrow k \text{ is any integer}$$

$$= \frac{1}{2} \sum_{k=-\infty}^{\infty} x(k) (z^{\frac{1}{2}})^{-k} + \frac{1}{2} \sum_{k=-\infty}^{\infty} x(k) (-z^{-\frac{1}{2}})^{-k}$$

$$= \frac{1}{2} X(z^{\frac{1}{2}}) + \frac{1}{2} X(-z^{\frac{1}{2}})$$

3.14.

(a) $X(z) = X_1(z)(1+3z^{-1})$

where $X_1(z) = \frac{1}{1+3z^{-1}+2z^{-2}} = \frac{1}{(1+z^{-1})(1+2z^{-1})}$

$$X_1(z) = \frac{1}{(1+z^{-1})(1+2z^{-1})} = \frac{z^2}{(z+1)(z+2)}$$

$$\frac{X_1(z)}{z} = \frac{z}{(z+1)(z+2)} = \frac{A_1}{z+1} + \frac{A_2}{z+2}$$

$$A_1 = \left.\frac{z}{z+2}\right|_{z=-1} = -1, \quad A_2 = \left.\frac{z}{z+1}\right|_{z=-2} = 2$$

Thus $X_1(z) = \frac{-1}{1+z^{-1}} + \frac{2}{1+2z^{-1}}$

$$x_1(n) = -(-1)^n u(n) + 2(-2)^n u(n)$$

Since $X(z) = X_1(z)(1+3z^{-1})$

we have $x(n) = x_1(n) + 3x_1(n-1) \Leftarrow \left\{ \begin{array}{l} \text{linearity} \\ \text{time-shifting} \end{array} \right.$

Thus
$$\begin{aligned} x(n) &= -(-1)^n u(n) + 2(-2)^n u(n) - 3(-1)^{n-1} u(n-1) \\ &\quad + 6(-2)^{n-1} u(n-1) \\ &= -(-1)^n [u(n) - u(n-1)] + 2(-1)^n u(n-1) \\ &\quad + 2(-2)^n [u(n) - u(n-1)] - (-2)^n u(n-1) \\ &= -(-1)^n \delta(n) + 2(-2)^n \delta(n) \\ &\quad + 2(-1)^n u(n-1) - (-2)^n u(n-1) \\ &= 2(-1)^n u(n) - (-2)^n u(n) \end{aligned}$$

Alternative method:

$$X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} = \frac{2}{1+z^{-1}} + \frac{-1}{1+2z^{-1}}$$

$$x(n) = [2(-1)^n - (-2)^n] u(n)$$

3.14

(i). $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}$

$= (1-\frac{1}{2}z^{-1})X_1(z)$, where $X_1(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}$

$x(n) = x_1(n) - \frac{1}{2}x_1(n-1). \Leftarrow \begin{cases}\text{linearity} \\ \text{time-shifting}\end{cases}$

Since $X_1(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}$ and $x_1(n)$ is causal

we have $x_1(n) = (-\frac{1}{2})^n u(n). \Leftarrow \left[a^n u(n) \overset{z}{\longleftrightarrow} \dfrac{1}{1-az^{-1}}\right]$

Thus $x(n) = (-\frac{1}{2})^n u(n) - \frac{1}{2}(-\frac{1}{2})^{n-1} u(n-1)$

$= (-\frac{1}{2})^n u(n) + (-\frac{1}{2})^n u(n-1)$

(j). $X(z) = \dfrac{1-az^{-1}}{z^{-1}-a} = (1-az^{-1})X_1(z)$, where $X_1(z) = \dfrac{1}{z^{-1}-a}$.

Thus $x(n) = x_1(n) - a x_1(n-1)$.

Since $X_1(z) = \dfrac{1}{z^{-1}-a} = \dfrac{-\frac{1}{a}}{1-\frac{1}{a}z^{-1}}$ and $x_1(n)$ is causal

we have $x_1(n) = -\frac{1}{a}(\frac{1}{a})^n u(n). = -(\frac{1}{a})^{n+1} u(n)$

Thus $x(n) = -(\frac{1}{a})^{n+1} u(n) + a(\frac{1}{a})^n u(n-1)$

$= -(\frac{1}{a})^{n+1} u(n) + (\frac{1}{a})^{n-1} u(n-1).$

3.23.

$$X(Z) = e^{z} + e^{\frac{1}{z}}, \quad |Z| \neq 0$$

Since $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x$

we have $X(Z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(\frac{1}{z})^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{Z^n}{n!} + \sum_{n=0}^{\infty} \frac{Z^{-n}}{n!}$$

$$= \sum_{n=-\infty}^{-1} \frac{Z^{-n}}{n!} + \sum_{n=0}^{\infty} \frac{Z^{-n}}{n!} + 1$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{n!}\right) Z^{-n} + 1$$

Thus $x(n) = \frac{1}{n!} + \delta(n)$

3.28

(1) Conjugation property: $x^*(n) \overset{Z}{\longleftrightarrow} X^*(z^*)$

Let $y(n) = x^*(n)$.

Then $Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$

$= \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$

$= \sum_{n=-\infty}^{\infty} \left[x(n)(z^{-n})^*\right]^*$

$= \left[\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n}\right]^*$

$= X^*(z^*)$

(2) Parseval's relation: $\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$

Since $x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$ (Eq.(3.1.16) in P157 Textbook)

we have $\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint_C X_1(z) z^{n-1} dz \, x_2^*(n)\right]$

$x_2^*(n)$ is constant with respect to the integration. $\Longrightarrow$ $= \frac{1}{2\pi j} \oint_C X_1(z) \left[\sum_{n=-\infty}^{\infty} x_2^*(n) \left(\frac{1}{z}\right)^{-n}\right] z^{n-1} dz$

$x^*(n) \overset{Z}{\longleftrightarrow} X^*(z^*)$ $\Longrightarrow$ $= \frac{1}{2\pi j} \oint_C X_1(z) X_2^*\left(\frac{1}{z^*}\right) dz$

3.32.

(a) $y(n) = 0.2y(n-1) + x(n) - 0.3x(n-1) + 0.02x(n-2)$.

$$Y(z) = 0.2z^{-1}Y(z) + X(z) - 0.3z^{-1}X(z) + 0.02z^{-2}X(z)$$

$$(1-0.2z^{-1})Y(z) = (1-0.3z^{-1}+0.02z^{-2})X(z)$$

$$H_1(z) = \frac{Y(z)}{X(z)} = \frac{1-0.3z^{-1}+0.02z^{-2}}{1-0.2z^{-1}}$$

$$= \frac{(1-0.1z^{-1})(1-0.2z^{-1})}{1-0.2z^{-1}}$$

$$= 1-0.1z^{-1}$$

(b) $y(n) = x(n) - 0.1x(n-1)$

$$Y(z) = X(z) - 0.1z^{-1}X(z) = (1-0.1z^{-1})X(z)$$

$$H_2(z) = \frac{Y(z)}{X(z)} = 1-0.1z^{-1}$$

Since $H_1(z) = H_2(z)$, the systems in (1) and (2) are equivalent.

3.40

Input: $x(n) = (\frac{1}{2})^n u(n) - \frac{1}{4}(\frac{1}{2})^{n-1} u(n-1)$

Output: $y(n) = (\frac{1}{3})^n u(n)$

Thus $X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{4}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$, $Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$

(a) $H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{1}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{3}{1-\frac{1}{4}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}}$

Thus $h(n) = [3(\frac{1}{4})^n - 2(\frac{1}{3})^n]\, u(n) \Leftarrow$ system is causal.

(b) $H(z) = \frac{1-\frac{1}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}}$

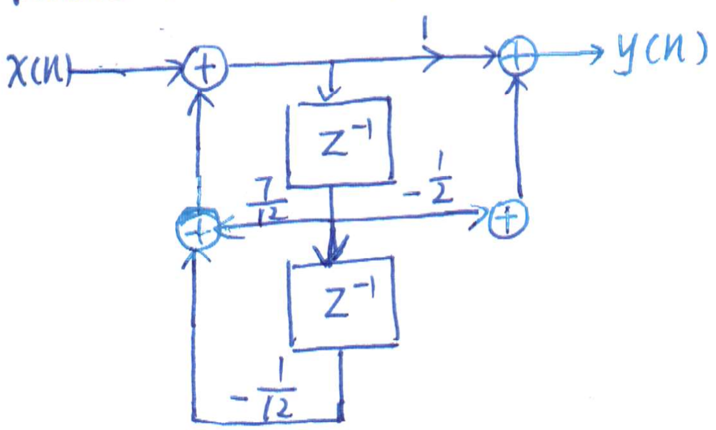
Thus $\frac{Y(z)}{X(z)} = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}}$

Thus $Y(z) - \frac{7}{12}z^{-1}Y(z) + \frac{1}{12}z^{-2}Y(z) = X(z) - \frac{1}{2}z^{-1}X(z)$

Thus the difference equation is

$y(n) = \frac{7}{12}y(n-1) - \frac{1}{12}y(n-2) + x(n) - \frac{1}{2}x(n-1)$

(c) Direct Form II Realization



(d) The poles of the system $p_1 = \frac{1}{3}$, $p_2 = \frac{1}{4}$ are both inside the unit circle. Thus the system is stable.

3.51.

$p_1 = -3$, $p_2 = -0.5$, $p_3 = 2$, $z = 1$.

(a) Thus $H(z) = \dfrac{k(z-1)}{(z+\frac{1}{2})(z+3)(z-2)}$, where k is any nonzero constant.

Since the system is stable, ROC has to include unit circle.
Therefore ROC is $\frac{1}{2} < |z| < 2$.

(b) If the system is causal, the ROC is the smallest circle encompassing all the poles, thus ROC is $|z| > 3$.
Since the system is causal and the poles $p_1 = -3$, $p_3 = 2$ lie outside of the unit circle, we have the system is not stable.

(c) (1) causal: ROC: $|z| > 3$

(2) anti-causal: ROC: $|z| < 3$

(3) noncausal: ROC: $\frac{1}{2} < |z| < 2$.

(4) noncausal: ROC: $2 < |z| < 3$.