4.4 Consider the following periodic signal:

$$x(n) = \{\ldots, 1, 0, 1, 2, \underset{\uparrow}{3}, 2, 1, 0, 1, \ldots\}$$

(a) sketch the signal $x(n)$ and its magnitude and phase spectra

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} = \frac{1}{6}\sum_{n=0}^{5} x(n)\, e^{-j2\pi kn/6}$$

$$= \frac{1}{6}\left[3 + 2e^{-j\cancel{2}\pi k/\cancel{6}3} + e^{-j2\pi k\cancel{2}/\cancel{6}3} + 0 + e^{-j\cancel{2}\pi k4/\cancel{6}3} + 2e^{-j\cancel{2}\pi k\frac{5}{\cancel{6}3}}\right]$$

$$= \frac{1}{6}\left[3 + 2\left(e^{-j\frac{\pi k}{3}} + e^{-j\frac{\pi k5}{3}}\right)^{+N} + \left(e^{-j\frac{2\pi k}{3}} + e^{-j\frac{4\pi k}{3}}\right)^{+2N}\right]$$

$$= \frac{1}{6}\left[3 + 4\cos\left(\frac{\pi k}{3}\right) + 2\cos\left(\frac{2\pi k}{3}\right)\right]$$

$$c_0 = \frac{1}{6}[3+4+2] = \frac{9}{6}$$

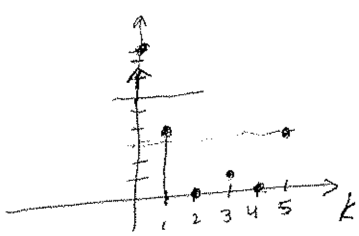
$$c_1 = \frac{1}{6}\left[3 + 4\cdot\frac{1}{2} + 2\cdot\left(\frac{-1}{2}\right)\right] = \frac{4}{6}$$

$$c_2 = \frac{1}{6}\left[4 + 4\left(\frac{-1}{2}\right) + 2\cdot\left(-\frac{1}{2}\right)\right] = 0$$

$$c_3 = \frac{1}{6}$$

$$c_4 = c_2 = 0$$

$$c_5 = c_1 = \frac{4}{6}$$



phase = 0

(b) using the results in part a, verify Parseval's relation by computing the power in the time and frequency domains.

$$P_{time} = \frac{1}{6} \sum_{n=0}^{5} |x(n)|^2$$

$$= \frac{1}{6} (3^2 + 2^2 + 1^2 + 0^2 + 1^2 + 2^2) = \frac{19}{6}$$

$$P_{freq} = \sum_{k=0}^{5} |c_k|^2 = \left[\left(\frac{9}{6}\right)^2 + \left(\frac{4}{6}\right)^2 + 0^2 + \left(\frac{1}{6}\right)^2 + 0^2 + \left(\frac{4}{6}\right)^2\right]$$

$$= \frac{19}{6}$$

$$P_{time} = P_{freq}. \quad \square$$

$\boxed{4.5}$ Consider the signal

$$x(n) = 2 + 2\cos\frac{\pi n}{4} + \cos\frac{\pi n}{2} + \frac{1}{2}\cos\frac{3\pi n}{4}$$

(a) Determine and sketch its power density spectrum.

$\underline{N=8}$

$$c_k = \frac{1}{8} \sum_{n=0}^{7} x(n)\, e^{-j2\pi kn/8}$$

$$x(n) = \{5.5, 3.0607, 1, 0.9393, 0.5, 0.9393, 1, 3.0607\}$$

or

$$= \left\{\frac{11}{2}, 2+\frac{3}{4}\sqrt{2}, 1, 2-\frac{3}{4}\sqrt{2}, \frac{1}{2}, 2-\frac{3}{4}\sqrt{2}, 1, 2+\frac{3}{4}\sqrt{2}\right\}.$$

$\Rightarrow c_0 = 2,\ c_1 = 1 = c_7,\ c_2 = 0.5,\ c_3 = \frac{1}{4} = c_5,\ c_4 = 0$

(b) power spectral density $\Rightarrow |c_k|^2$

~~4.6~~.

$\boxed{4.5}$ (b) Evaluate the power of the signal.

$$P = \sum_{i=0}^{7} |c(i)|^2 = 4 + 1 + 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \frac{1}{16} = \frac{53}{8}$$

4.6 Determine and sketch the magnitude and phase spectra of the following periodic signals.

(b) $x(n) = \cos \frac{2\pi}{3} n + \sin \frac{2\pi}{5} n$

(15-5)

$\Rightarrow N = 15$

$\cos \frac{2\pi}{3} n = \frac{1}{2}\left[e^{j\frac{2\pi}{3}n} + e^{-j\frac{2\pi}{3}n}\right]$

$$C_{k1} = \begin{cases} \frac{1}{2} & , k = 5, 10 \\ 0 & , o/w \end{cases}$$

$\sin \frac{2\pi}{5} n = \frac{1}{2j}\left[e^{\frac{j2\pi n}{5}} - e^{-j\frac{2\pi n}{5}}\right]$

$$C_{k_2} = \begin{cases} \frac{1}{2j} & , k = 3 \\ \frac{-1}{2j} & , k = 12 \\ 0 & , o/w \end{cases}$$

$$\Rightarrow C_{k\,(\text{overall})} = C_{k_1} + C_{k_2} = \begin{cases} \frac{1}{2j} & , k = 3 \\ \frac{1}{2} & , k = 5 \\ \frac{1}{2} & , k = 10 \\ \frac{-1}{2j} & , k = 12 \\ 0 & , o/w \end{cases}$$

phase

$$\begin{cases} \frac{3\pi}{2} & , k = 3 \\ 0 & , k = 5 \\ 0 & , k = 10 \\ \pi/2 & , k = 12 \\ 0 & , o/w \end{cases}$$

(d) $x(n) = \{ \ldots, -2, -1, \underset{\uparrow}{0}, 1, 2, -2, -1, 0, 1, 2 \} \ldots \}$

$\Rightarrow \boxed{N=5}$

$$C_k = \frac{1}{5} \sum_{n=0}^{4} x(n)\, e^{-j\frac{2\pi n k}{5}}$$

$$C_k = \frac{1}{5}\left[1.\, e^{-j\frac{2\pi k}{5}} + 2.\, e^{-j\frac{2\pi.2.k}{5}} - 2\, e^{-j\frac{2\pi}{5}.3.k} - e^{-j\frac{2\pi}{5}.4.k}\right]$$

$$= \frac{1}{5}\left[\left(e^{-j\frac{2\pi k}{5}} - e^{-j\frac{8\pi}{5}k + 2N}\right) + 2\left(e^{-j\frac{4\pi}{5}k} - e^{-j\frac{6}{5}\pi k + 10}\right)\right]$$

$$= \frac{-2j}{5}\left[\sin\left(\frac{2\pi k}{5}\right) + 2.\sin\left(\frac{4\pi k}{5}\right)\right]$$

phase

0

$\pi + \frac{\pi}{2} + \angle[\ \ ]$

$C_0 = 0$,

$C_1 = \frac{-2j}{5}\left[+\sin\left(\frac{2\pi}{5}\right) + 2\sin\left(\frac{4\pi}{5}\right)\right]$

$C_2 = \frac{-2j}{5}\left[\sin\left(\frac{4\pi}{5}\right) + 2\sin\left(\frac{8\pi}{5}\right)\right]$

$C_3 = -C_2$

$C_4 = -C_1$

(f) $x(n) = \{ \ldots, 0, 0, \underset{\uparrow}{1}, 1, 0, 0, 0, 1, 1, 0, 0, \ldots \}$

$N = 5$

$$C_k = \frac{1}{5}\sum_{n=0}^{4} x(n)\, e^{-j\frac{2\pi k n}{5}} = \frac{1}{5}\left[1 + e^{-j\frac{2\pi k}{5}}\right] \times \frac{e^{j\frac{\pi k}{5}}}{e^{j\frac{\pi k}{5}}}$$

$$= \frac{2}{5}\cos\left(\frac{\pi k}{5}\right) e^{-j\frac{\pi k}{5}}$$

$\rightarrow C_0 = \frac{2}{5}$, $C_1 = \frac{2}{5}\cos\left(\frac{\pi}{5}\right) e^{-j\frac{\pi}{5}}$

$C_2 = \frac{2}{5}\cos\left(\frac{2\pi}{5}\right) e^{-j\frac{2\pi}{5}}$, $C_3 = \frac{2}{5}\cos\left(\frac{3\pi}{5}\right) e^{-j\frac{3\pi}{5}}$, $C_4 = \frac{2}{5}\cos\left(\frac{4\pi}{5}\right) e^{-j\frac{4\pi}{5}}$

$\boxed{4.7}$ Determine the periodic signal $x(n)$, with fundamental period $N=8$, if the Fourier coefficients are given by.

(a) $c_k = \cos\frac{\pi k}{4} + \sin\frac{3\pi k}{4} \Rightarrow c_k = \frac{1}{2}\left[e^{j\frac{2\pi k}{8}} + e^{-j\frac{2\pi k}{8}}\right] + \frac{1}{2j}\left[e^{j\frac{6\pi k}{8}} - e^{-j\frac{6\pi k}{8}}\right]$

$$x(n) = \sum_{k=0}^{7} c_k e^{j\frac{2\pi nk}{8}}$$

if $c_k = e^{j\frac{2\pi pk}{8}} \Rightarrow \sum_{k=0}^{7} e^{j\frac{2\pi pk}{8}} e^{j\frac{2\pi nk}{8}} = \begin{cases} 8 & , n=-p \\ 0 & , n\neq -p \end{cases}$

$$\Rightarrow x(n) = 4\,\delta(n+1) + 4\,\delta(n-1) + 4j\,\delta(n+3) + 4j\,\delta(n-3)$$

$\boxed{4.8}$ Two DT signals, $s_k(n)$ and $s_\ell(n)$, are said to be orthogonal over an interval $[N_1, N_2]$ if

$$\sum_{n=N_1}^{N_2} s_k(n)\, s_\ell^*(n) = \begin{cases} A_k & , k=\ell \\ 0 & , k\neq \ell \end{cases}$$

if $A_k = 1$, the signals are orthonormal.

(a) Prove the relation

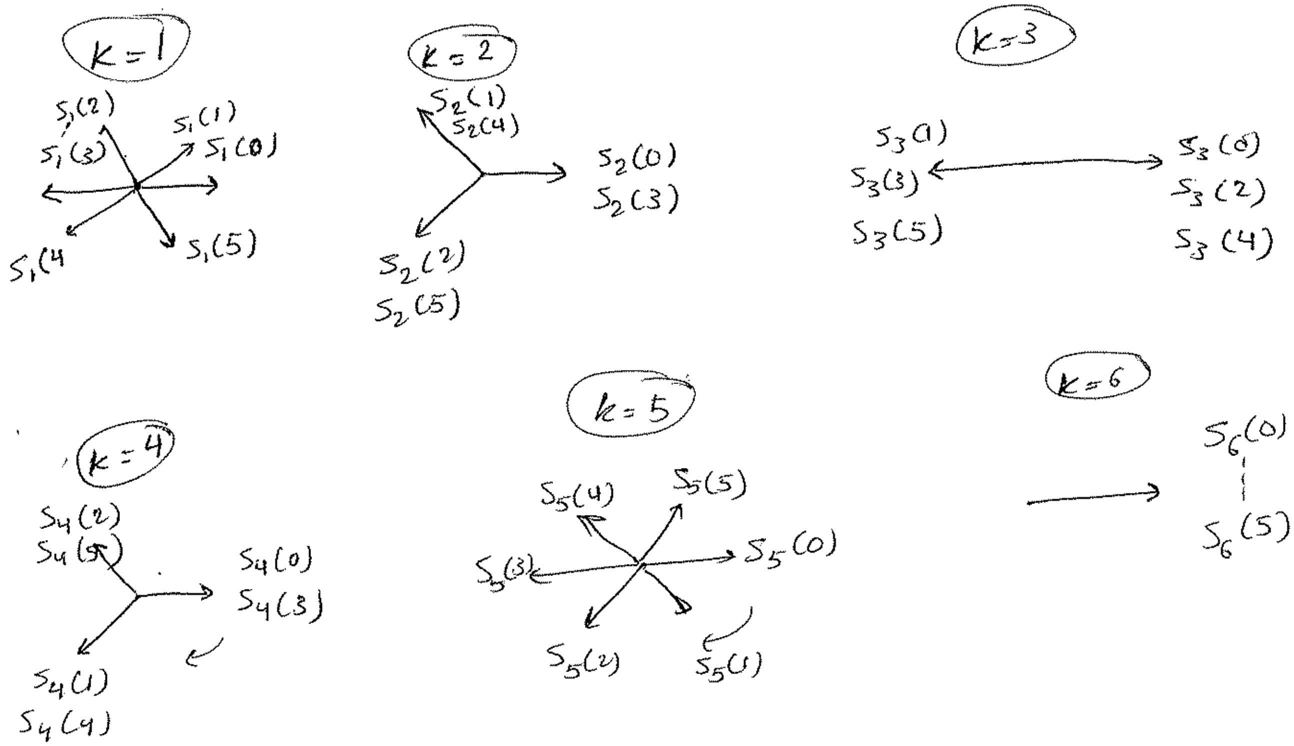
$$\sum_{n=0}^{N-1} e^{j\frac{2\pi kn}{N}} = \begin{cases} N & , k=0, \mp N, \mp 2N, \ldots \\ 0 & , o/w. \end{cases}$$

if $k=0, \mp N, \mp 2N, \ldots$

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} = \sum_{n=0}^{N-1} \overbrace{\cos\left(\frac{2\pi kn}{N}\right)}^{1} - \overbrace{\cancel{\sin\left(\frac{2\pi kn}{N}\right)}}^{0} = N$$

for $k \neq 0, \mp N, \mp 2N, \ldots \Rightarrow \sum_{n=0}^{N-1} e^{j2\pi kn/N} = \frac{1-e^{j2\pi k}}{1-e^{j2\pi k/N}} = 0$

(b) Illustrate the validity of the relation in part (a) by plotting for every value of $k = 1, 2, \ldots, 6$, the signals $S_k = e^{j(\frac{2\pi}{6})kn}$ $n = 0, 1, \ldots, 5$. [Note: for a given k, n the signal $S_k(n)$ can be represented as a vector in the complex plane.]



(c) Show that the harmonically related signals

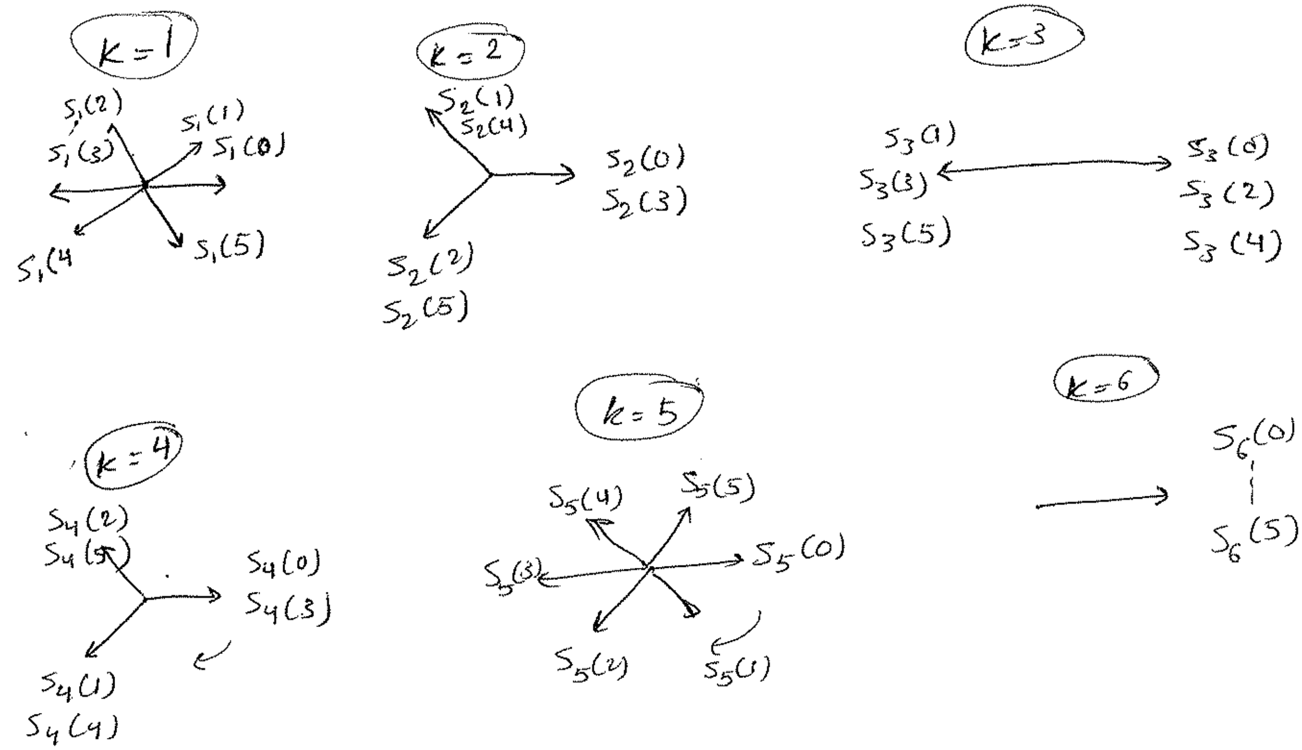
$$S_k(n) = e^{j(2\pi/N)kn}$$

are orthogonal over any interval of length N.

$$\sum_{n=0}^{N-1} S_k(n)\, S_i^*(n) = \sum_{n=0}^{N-1} e^{j2\pi kn/N} e^{-j2\pi in/N}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi(k-i)n}{N}} = \begin{cases} N, & k = i \\ 0, & k \neq i \end{cases}$$

(b) Illustrate the validity of the relation in part (a) by plotting for every value of $k = 1, 2, \ldots, 6$, the signals $S_k = e^{j\left(\frac{2\pi}{6}\right)kn}$, $n = 0, 1, \ldots, 5$. [Note: for a given k, n the signal $S_k(n)$ can be represented as a vector in the complex plane.]



(c) Show that the harmonically related signals

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$$= \sum_{n=0}^{N-1} e^{\frac{j2\pi(k-i)n}{N}} = \begin{cases} N, & k = i \\ 0, & k \neq i \end{cases}$$

$\boxed{4.10}$ Determine the signals having the following Fourier Transform

(b) $X(\omega) = \cos^2 \omega = \left(\frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega}\right)^2$

$$= \frac{1}{4}\left(e^{j2\omega} + 2 + e^{-j2\omega}\right)$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{8\pi}\left[2\pi\delta(n+2) + 4\pi\,\delta(n) + 2\pi\,\delta(n-2)\right]$$

$$= \frac{1}{4}\left[\delta(n+2) + 2\delta(n) + \delta(n-2)\right]$$

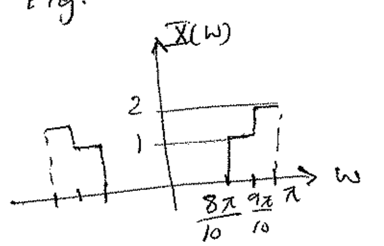
(c) $X(\omega) = \begin{cases} 1, & \omega_0 - \delta\omega/2 \leq |\omega| \leq \omega_0 + \delta\omega/2 \\ 0, & \text{elsewhere} \end{cases}$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{\omega_0 - \frac{\delta\omega}{2}}^{\omega_0 + \frac{\delta\omega}{2}} e^{j\omega n}\, d\omega$$

$$= \frac{2}{\pi}\,\delta\omega\left(\frac{\sin(n\delta\omega/2)}{n\delta\omega/2}\right) e^{jn\omega_0}$$

$\boxed{4.12}$ Determine the signal $x(n)$ if its Fourier transform is as given in Fig.

(a)



$$x(n) = \frac{1}{2\pi}\left[\int_{8\pi/10}^{9\pi/10} e^{j\omega n}\, d\omega + \int_{-9\pi/10}^{-8\pi/10} e^{j\omega n}\, d\omega + 2\int_{\frac{9\pi}{10}}^{\pi} e^{j\omega n}\, d\omega + 2\int_{-\pi}^{-\frac{9\pi}{10}} e^{j\omega n}\, d\omega\right]$$

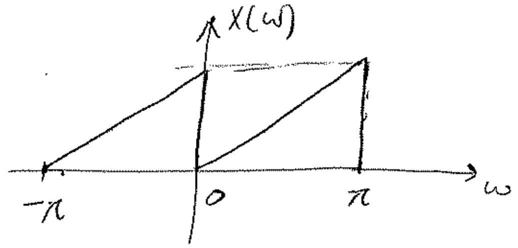
$$= \frac{1}{2\pi}\left[\frac{1}{jn}\left(e^{j\frac{9\pi n}{10}} - e^{-j\frac{9\pi n}{10}} - e^{j\frac{8\pi n}{10}} + e^{-j\frac{8\pi n}{10}}\right)\right.$$

$$\left. + \frac{2}{jn}\left(-e^{j9\pi n/10} + e^{-j\frac{9\pi n}{10}} + e^{j\pi n} - e^{-j\pi n}\right)\right]$$

$$= \frac{1}{n\pi}\left[\cancelto{0}{\sin \pi n} - \sin\frac{8\pi n}{10} - \sin\frac{9\pi n}{10}\right]$$

$=$

(b)



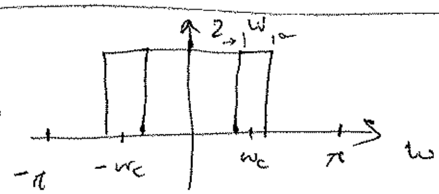
$$X(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{0} X(w)e^{jwn}dw + \int_{0}^{\pi} X(w)e^{jwn}dw\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(\frac{w}{\pi}+1\right)e^{jwn}dw + \int_{0}^{\pi}\frac{w}{\pi}e^{jwn}dw\right]$$

$$= \frac{1}{2\pi}\left[\frac{w}{jn\pi}e^{jwn}\Big|_{-\pi}^{\pi} + \frac{e^{jwn}}{jn}\Big|_{-\pi}^{0}\right]$$

$$= \frac{1}{\pi n}\sin\frac{\pi n}{2}\, e^{-jn\pi/2}$$

(c)



$$X(n) = \frac{1}{2\pi}\int_{w_c-\frac{w}{2}}^{w_c+\frac{w}{2}} 2e^{jwn}dw + \frac{1}{2\pi}\int_{-w_c-\frac{w}{2}}^{-w_c+\frac{w}{2}} 2e^{jwn}dw$$

$$= \frac{1}{\pi}\left[\frac{1}{jn\pi}e^{jwn}\Big|_{w_c-\frac{w}{2}}^{w_c+\frac{w}{2}} + \frac{e^{jwn}}{jn}\Big|_{-w_c-\frac{w}{2}}^{-w_c+\frac{w}{2}}\right]$$

$$X(n) = \frac{2}{\pi n}\left[\frac{e^{j(w_c+\frac{w}{2})n} - e^{j(w_c-\frac{w}{2})n} + e^{-j(w_c-\frac{w}{2})n} - e^{-j(w_c+\frac{w}{2})n}}{2j}\right]$$

$$= \frac{2}{\pi n}\left[\sin\left(w_c+\frac{w}{2}\right)n - \sin\left(w_c-\frac{w}{2}\right)n\right]$$

4.14 Consider the signal

$$x(n) = \{-1, 2, \underset{\uparrow}{-3}, 2, -1\}$$

with Fourier transform $X(w)$, Compute the following quantities, without explicitly computing $X(w)$.

a) $X(0) = \sum_n x(n)e^{\cancel{jwn}^{\,0}} = -1$

b) $\angle X(w) = \tan^{-1}\frac{X_I(w)}{X_R(w)} = \tan^{-1}0$ (real, even)

$= \pi$ for all w.

hint $\left(\tan^{-1}\frac{0}{-1}\right) \to \pi$

(c) $\int_{-\pi}^{\pi} X(\omega)\,d\omega$

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\,d\omega, \quad \int_{-\pi}^{\pi} X(\omega)\,d\omega = 2\pi x(0) = -6$$

(d) $$X(\pi) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-jn\pi} = \sum_n (-1)^n x(n) = \cancel{\pi} -3-2-2-1-1 = -9$$

(e) $$\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = 2\pi \sum_n |x(n)|^2 = 2\pi(9+8+2) = 38\pi$$

$\boxed{4.17}$ Let $x(n)$ be an arbitrary signal, not necessarily real valued with Fourier transform $X(\omega)$. Express the Fourier transforms of the following signals in terms of $X(\omega)$.

a) $$x^*(n) \Rightarrow \sum_n x^*(+n)\,e^{-j\omega n} = \Big(\sum_n x(n)\,e^{-j(-\omega)n}\Big)^* = X^*(-\omega)$$

b) $$x^*(-n) \Rightarrow \sum_n x^*(-n)\,e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n)^*\,e^{+j\omega n} = X^*(\omega)$$

c) $y(n) = x(n) - x(n-1)$.

$$\sum_n y(n)\,e^{-j\omega n} = \sum_n x(n)\,e^{-j\omega n} - \sum_n x(n-1)\,e^{-j\omega n}$$

$$Y(\omega) = X(\omega) + X(\omega)\,e^{-j\omega}$$

$$= (1-e^{-j\omega})\,X(\omega).$$

(d) $$y(n) = \sum_{k=-\infty}^{n} x(k).$$

$$y(n) = \cancel{y(n) - y(n-1)} \underbrace{\sum_{-\infty}^{n-1} x(k)}_{y(n-1)} + x(n).$$

$$x(n) = y(n) - y(n-1)$$

$$X(\omega) = Y(\omega) - Y(\omega)\,e^{-j\omega}$$

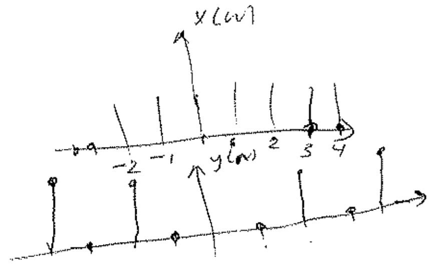
$$X(\omega) = Y(\omega)\,(1-e^{-j\omega})$$

$$\Rightarrow Y(\omega) = \frac{X(\omega)}{1-e^{-j\omega}}$$

(e) $y(n) = x(2n)$

$$Y(\omega) = \sum_n x(2n)\, e^{-j\omega n}$$
$$= \sum_n x(n)\, e^{-j\frac{\omega}{2}n}$$
$$= X\left(\frac{\omega}{2}\right)$$

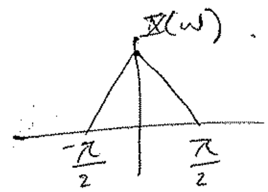
(f) $y(n) = \begin{cases} x(n/2), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$



$$Y(\omega) = \sum_n x\left(\frac{n}{2}\right) e^{-j\omega n}$$
$$= \sum_n x(n)\, e^{-j(2\omega)n}$$
$$= X(2\omega)$$

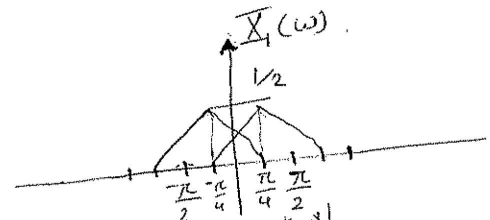
~~down sampling~~.

4.19 Let $x(n)$ be a signal with Fourier transform as in Fig. P4.19. Determine and sketch the Fourier transform of the following signals.



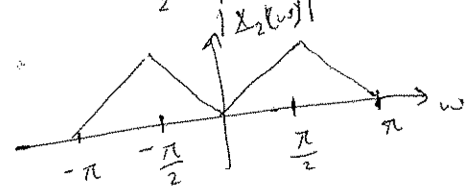
(a) $x_1(n) = x(n)\cos(\pi n/4)$

$$x_1(n) = x(n)\cdot\frac{1}{2}\left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right)$$
$$X_1(\omega) = \frac{1}{2}\left[X\left(\omega-\frac{\pi}{4}\right) + X\left(\omega+\frac{\pi}{4}\right)\right]$$



(b) $x_2(n) = x(n)\sin\left(\frac{\pi n}{2}\right)$

$$X_2(\omega) = \frac{1}{2j}\left[X\left(\omega-\frac{\pi}{2}\right) + X\left(\omega+\frac{\pi}{2}\right)\right]$$

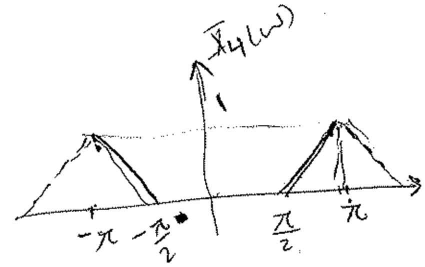


(c) $x_3(n) = x(n)\cos\left(\frac{\pi n}{2}\right)$

$$X_3(\omega) = \frac{1}{2}\left[X\left(\omega-\frac{\pi}{2}\right) + X\left(\omega+\frac{\pi}{2}\right)\right]$$

(d) $x_4(n) = x(n)\cos \pi n$

$$X_4(\omega) = \frac{1}{2}\left[X(\omega-\pi) + X(\omega+\pi)\right]$$
$$= X(\omega-\pi)$$



4.22 A Signal $x(n)$ has the following Fourier transform

$$X(\omega) = \frac{1}{1-ae^{-j\omega}}$$

Determine the Fourier transforms of the following signals.

(a) $x(2n+1)$

$$X_1(\omega) = \sum_n x(2n+1)\, e^{-j\omega n}$$

$$= \sum_k x(k)\, e^{-j\frac{\omega k}{2}}\, e^{j\frac{\omega}{2}}$$

$$= X\left(\frac{\omega}{2}\right) e^{j\omega/2}$$

$$= \frac{e^{j\frac{\omega}{2}}}{1-ae^{-j\frac{\omega}{2}}}$$

(b) $e^{\pi n/2}\, x(n+2)$

$$X_2(\omega) = \sum_n x(n+2)\, e^{\pi n/2}\, e^{-j\omega n} =$$

$$= -\sum_k x(k)\, e^{-jk(\omega+j\pi/2)}\, e^{j2\omega}$$

$$= -X\left(\omega+\frac{j\pi}{2}\right) e^{j2\omega}$$

(c) $x(-2n)$

$K=-2n$

$n=-\frac{k}{2}$

$$X_3(\omega) = \sum_n x(-2n)\, e^{-j\omega n} = +\sum_k x(k)\, e^{-jk(\omega/2)}$$

$$= X\left(-\frac{\omega}{2}\right)$$

(d) $x(n)\cos(0.3\pi n)$

$$X_4(\omega) = \sum_n \frac{1}{2}\left(e^{j0.3\pi n} + e^{-j0.3\pi n}\right) x(n)\, e^{-j\omega n}$$

$$= \frac{1}{2}\sum_n x(n)\left[e^{-j(\omega-0.3\pi)n} + e^{-j(\omega+0.3\pi)n}\right]$$

$$= \frac{1}{2}\left[X(\omega-0.3\pi) + X(\omega+0.3\pi)\right]$$

(e) $x(n) \circledast x(n-1)$

$$X_5(\omega) = X(\omega)\left[X(\omega)e^{-j\omega}\right] = X^2(\omega)e^{-j\omega}$$

(f) $x(n) \circledast x(-n)$

$$X_6(\omega) = X(\omega)\,X(-\omega)$$

$$= \frac{1}{(1-ae^{-j\omega})(1-ae^{j\omega})}$$

$$= \frac{1}{(1-2a\cos\omega+a^2)}$$