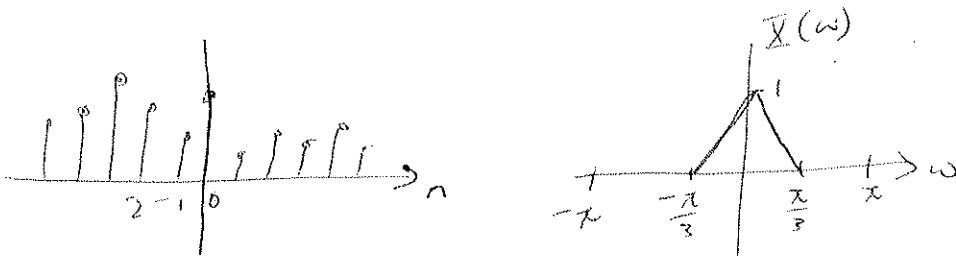


11.5 In this problem we illustrate the concepts of sampling and decimation for discrete-time signals. To this end consider a signal $x(n]$ with Fourier transform $X(\omega)$ as in



(a) Sampling $x(n]$ with a sampling period $D=2$, results in the signal

$$X_s(n) = \begin{cases} x(n), & n = 0, \pm 2, \pm 4, \dots \\ 0, & n = \pm 1, \pm 3, \pm 5, \dots \end{cases}$$

compute and sketch the signal $X_s(n]$ and its Fourier transform $X_s(\omega)$. Can we reconstruct $x(\omega)$ from $X_s(n]$? How?

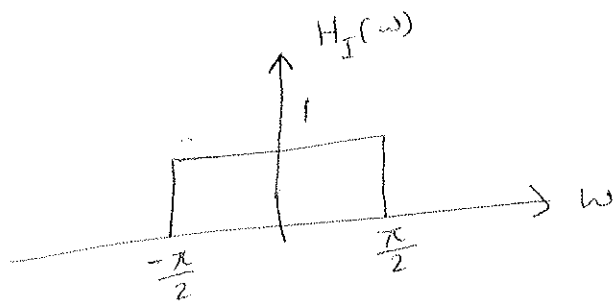
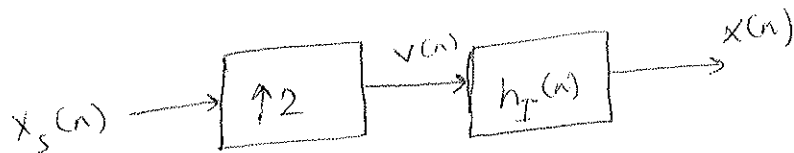
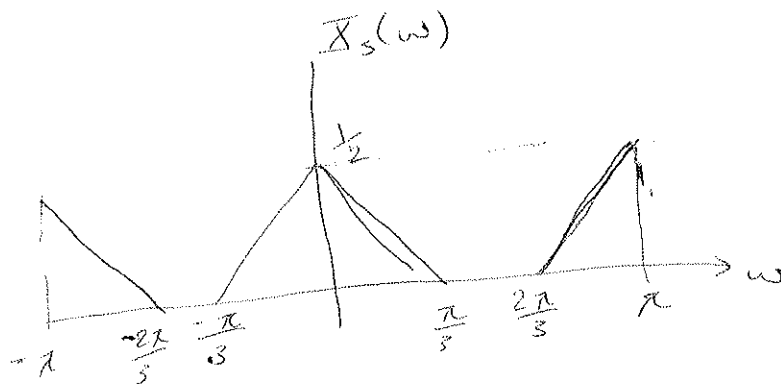
$$\text{let } \omega = \frac{\omega'}{F_x}, \quad \omega' = \frac{2\omega'}{F_x}$$

$$X_s(\omega) = \sum_n x_s(n) e^{-j\omega n} = \sum_n x(2m) e^{-j\omega 2m}$$

$$= \frac{1}{2} \sum_q X(\omega - \frac{2\pi q}{2})$$

$$= \frac{1}{2} \sum_q X(\omega - \pi q)$$

to recover $x(n]$ from $X_s(n]$



b) Decimating $x(n)$ by a factor of $D=2$, produces the signal

$$x_d(n) = x(2n), \quad \text{all } n$$

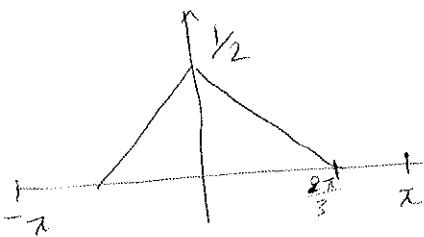
show that $\bar{X}_d(\omega) = \bar{X}_s(\frac{\omega}{2})$. plot the signal $x_d(n)$ and its transform $\bar{X}_d(\omega)$.

Do we lose any information when we decimate the sampled signal $x_s(n)$?

recall $\omega' = 2\omega$

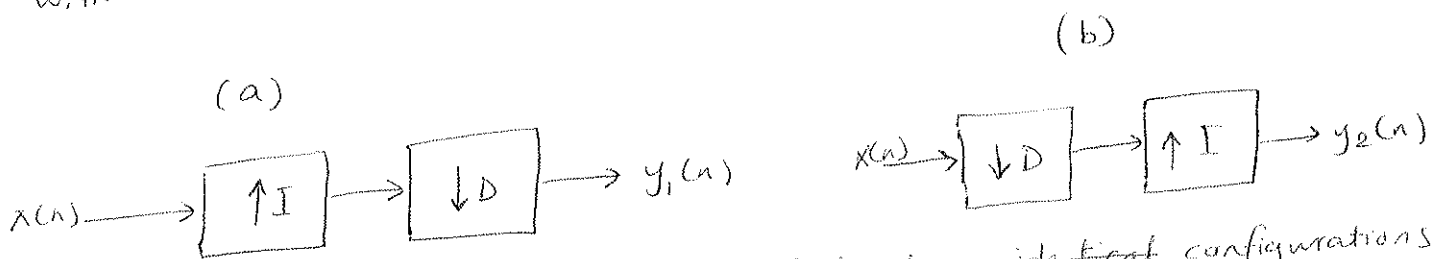
$$\begin{aligned} \bar{X}_d(\omega') &= \sum_n x_d(n) e^{-j\omega'n} \\ &= \sum_{\text{even}} x_s(n) e^{-j\omega' \frac{n}{2}} \\ &= \sum_n x_s(n) e^{-j\omega' \frac{n}{2}} \\ &= \bar{X}_s\left(\frac{\omega'}{2}\right) \quad \square \end{aligned}$$

since $x_s(n) = 0$ for odd n .



No information is lost, since the decimated sample rate still exceeds twice the band limit of the original sig.

11.9 Consider the two different ways of cascading a decimator with an interpolator shown in



(a) if $D=I$, show that the outputs of the two identical configurations are different. Hence, in general, the two systems are not identical.

let $x(n) = \{x_0, x_1, \dots\}$

$D=I=2$, decimation first (fig-b)

$$z_2(n) = \{x_0, x_2, x_4, \dots\}$$

$$y_2(n) = \{x_0, 0, x_2, 0, x_4, 0, \dots\}$$

Interpolation first (fig-a)

$$z_1(n) = \{x_0, 0, x_1, 0, x_2, 0, \dots\}$$

$$y_1(n) = \{x_0, x_1, x_2, \dots\}$$

$$\Rightarrow y_2(n) \neq y_1(n)$$

(b) show that the two systems are identical if and only if D & I are relatively prime.

let $D = dk$ and $I = ik$ and d, i are relatively prime.

$$x(n) = \{x_0, x_1, x_2, \dots\}$$

decimation first

$$z_2(n) = \{x_0, x_{dk}, x_{2dk}, \dots\}$$

$$y_2(n) = \left\{ x_0, \underbrace{0, 0, \dots, 0}_{i k - 1}, x_{dk}, \underbrace{0, \dots, 0}_{i k - 1}, x_{2dk}, \dots \right\}$$

interpolation first

$$z_1(n) = \{ \underbrace{x_0, 0, 0, \dots, 0}_{ik-1}, \underbrace{x_1, 0, 0, \dots, 0}_{ik-1}, x_2, 0, \dots, 0, \dots \}$$

$$y_1(n) = \{ \underbrace{x_0, 0, 0, \dots, 0}_{d-1}, \underbrace{x_d, 0, \dots, 0}_{d-1}, \dots \}$$

$\Rightarrow y_1 = y_2$ iff $d = dk$ or $k=1$, $\Rightarrow D$ and I are relatively prime

example let $x_n = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, \dots\}$

$$D=2, I=3$$

Decimation First:

$$z_1(n) = \{x_1, x_3, x_5, \dots\}$$

$$y_1(n) = \{ \underbrace{x_1, 0, 0}_{I-1}, \underbrace{x_3, 0, 0}_{I-1}, x_5, \dots \}$$

Interpolation first:

$$z_2(n) = \{x_1, \underbrace{0, 0}_{I-1}, x_2, \underbrace{0, 0}_{I-1}, \underbrace{x_3, 0, 0}_{I-1}, \underbrace{x_4, 0, 0}_{I-1}, x_5, \dots\}$$

$$y_2(n) = \{x_1, \underbrace{0, 0}_{I-1}, x_3, \underbrace{0, 0}_{I-1}, x_5, \dots\}$$

$$\Rightarrow y_1(n) = y_2(n)$$

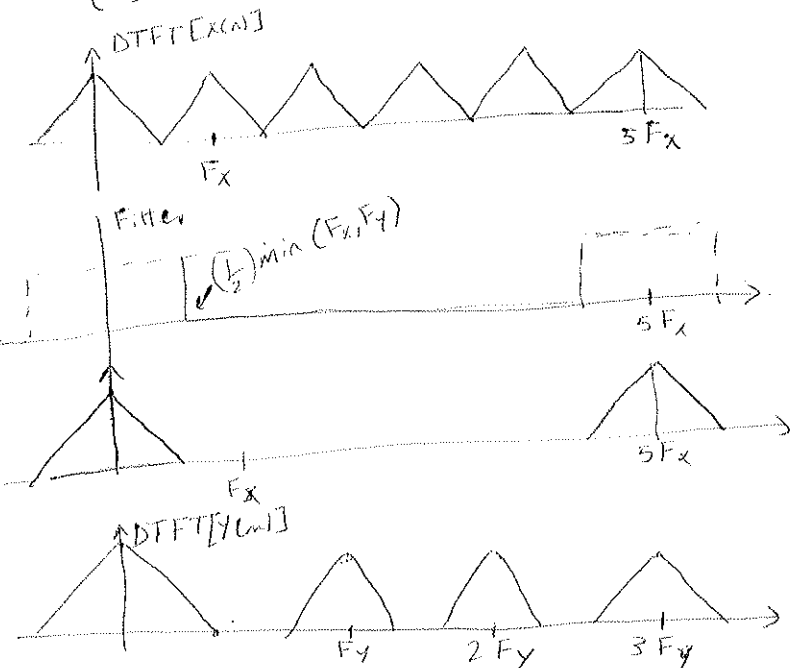
11.12 A sequence $x(n)$ is upsampled by $I=2$, it passes through an LTI system $H_1(z)$, and then it is downsampled by $D=2$, can we replace this process with a single LTI system $H_2(z)$? if the answer is positive, determine the system function of this system.

output of upsampler is $X(z^2)$

$$\begin{aligned}
 Y_1(z) &= \frac{1}{2} [X(z) H_1(z^{1/2}) + X(z) H_1(z^{1/2} W^{1/2})] \\
 &= \frac{1}{2} [H_1(z^{1/2}) + H_1(z^{1/2} W^{1/2})] X(z) \\
 &= H_2(z) X(z)
 \end{aligned}$$

11.13 Plot the signals and their corresponding spectra for rational sampling rate conversion by (a) $I/D = 5/3$
 (b) $I/D = 3/5$.
 Assume that the spectrum of the input signal $x(n)$ occupies the entire range $-\pi \leq \omega_x \leq \pi$

(a) $I/D = 5/3$



(b) $1/D = 3/5$

