Properties of the Fourier Transform for Discrete-Time Signals

Notation

- Direct Transform (Analysis)

$$X(\omega) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad (1)$$

- Inverse Transform (Synthesis)

$$x(n) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega \quad (2)$$

- Fourier Transform pair

$$x(n) \xleftrightarrow{\text{F}} X(\omega)$$

Symmetry Properties of the Fourier Transform

- Exploiting symmetry to arrive at simpler formulas for both the direct and inverse Fourier transform.

Suppose $x(n)$ and $X(\omega)$ are complex-valued functions.

$$\begin{aligned} x(n) &= x_R(n) + jx_I(n) \\ X(\omega) &= X_R(\omega) + jX_I(\omega) \end{aligned} \quad (3)$$

using (remember $\cos -\omega = \cos \omega$ and $\sin -\omega = -\sin \omega$)

$$e^{-j\omega} = \cos \omega - j \sin \omega$$

Symmetry Properties of the Fourier Transform-cont

into equation 1

$$X_R(\omega) + jX_I(\omega) = \sum_{n=-\infty}^{\infty} [x_R(n) + jx_I(n)] [\cos \omega n - j \sin \omega n]$$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} [x_R(n) \cos \omega n + x_I(n) \sin \omega n]$$

$$X_I(\omega) = - \sum_{n=-\infty}^{\infty} [x_R(n) \sin \omega n - x_I(n) \cos \omega n]$$

Similarly with equation 1, we get

$$x_R(n) = \frac{1}{2\pi} \int_{2\pi} [X_R(\omega) \cos \omega n - X_I \sin \omega n] \, d\omega$$

$$x_I(n) = \frac{1}{2\pi} \int_{2\pi} [X_R(\omega) \sin \omega n + X_I \cos \omega n] \, d\omega$$

Special cases: Real signals

For a real $x(n) \Rightarrow x_I(n) = 0$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cos \omega n$$

$$X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n) \sin \omega n$$

Recall $\cos(-\omega n) = \cos \omega n$ and $\sin(-\omega n) = -\sin \omega n$ we get:

$$X_R(-\omega) = X_R(\omega), (even)$$
$$X_I(-\omega) = -X_I(\omega), (odd)$$
$$\Rightarrow X^*(\omega) = X(-\omega)$$

- The spectrum of a real signal has Hermitian symmetry.

Real Signals, cont.

- Magnitude and phase spectra for real signals

$$|X(\omega)| = \sqrt{X_R^2(\omega) + X_I^2(\omega)}, (even)$$

$$\angle X(\omega) = \tan^{-1} \frac{X_I(\omega)}{X_R(\omega)}, \text{ (odd)}$$

- Inverse transform of a real signal, $x(n) = x_R(n)$ implies

$$x(n) = \frac{1}{2\pi} \int_{2\pi} [X_R(\omega) \cos \omega n - X_I(\omega) \sin \omega n] \, d\omega$$

$$x(n) = \frac{1}{\pi} \int_{0}^{\pi} [X_R(\omega) \cos \omega n - X_I(\omega) \sin \omega n] \, d\omega$$

Special cases: Real and even signals

- If $x(n)$ is real and even $x(-n) = x(n)$, then $x(n)\cos(\omega n)$ is even and $x(n)\sin(\omega n)$ is odd

$$X_R(\omega) = x(0) + 2\sum_{n=1}^{\infty} x(n)\cos\omega n, (even)$$

$$X_I(\omega) = 0$$

$$x(n) = \frac{1}{\pi}\int_0^{\pi} X_R(\omega)\cos\omega n d\omega$$

- real and even signals have a real-valued spectra, which is also even in ω.

Example 4.4.2

- Determine the Fourier transform of the signal

$$x(n) = \begin{cases} A, -M \leq n \leq M \\ 0, elsewhere \end{cases}$$

solution

Notice that $x(-n) = x(n) \Rightarrow x(n)$ real and even.

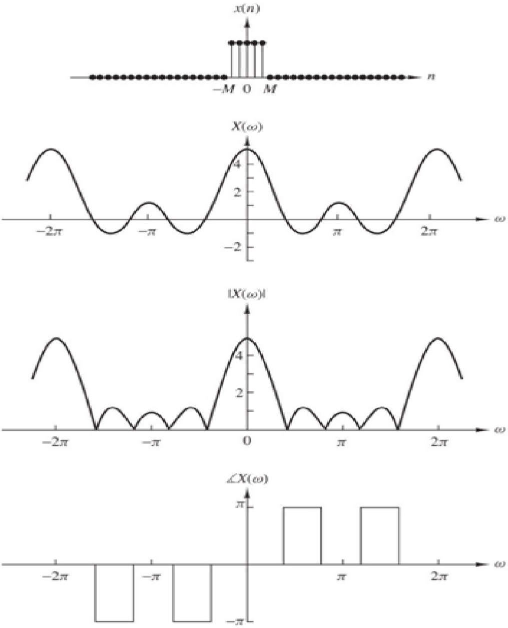
$$X(\omega) = X_R(\omega) = A\left(1 + 2\sum_{n=1}^{M} \cos \omega n\right)$$

Since $X(\omega)$ is real,

$$|X(\omega)| = \left|A\left(1 + 2\sum_{n=1}^{M} \cos \omega n\right)\right|$$

Example, cont.

$$Phase X(\omega) = \begin{cases} 0, X(\omega) > 0 \\ \pi, X(\omega) < 0 \end{cases}$$


x(n)
n
−M
0
M
X(ω)
4
2
−2
ω
−2π
−π
π
2π
|X(ω)|
4
2
ω
−2π
−π
0
π
2π
∠X(ω)
π
−π
ω
−2π
−π
π
2π

Example: linearity

- Determine the Fourier transform of the signal

$$x(n) = a^{|n|}, -1 < a < 1$$

solution

Let $x(n) = x_1(n) + x_2(n)$ where

$$x_1(n) = \begin{cases} a^n, n \geq 0 \\ 0, n < 0 \end{cases}$$

and

$$x_2(n) = \begin{cases} a^{-n}, n < 0 \\ 0, n \geq 0 \end{cases}$$

example, cont.

we are going to use the linearity property $X(\omega) = X_1(\omega) + X_2(\omega)$ as follows:

$$X_1(\omega) = \sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n} = \sum_{0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty}(ae^{-j\omega})^n$$

Using geometric series rule for $|ae^{-j\omega n}| < 1$

$$X_1(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

example, cont

Similarly, for $x_2(n)$

$$X_2(\omega) = \sum_{n=-\infty}^{\infty} x_2(n)e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n}e^{-j\omega n}$$
$$= \sum_{n=-\infty}^{-1} ae^{j\omega^{-n}} = \sum_{k=1}^{\infty}(ae^{j}\omega)^{k}$$
$$= \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

Thus,

$$X(\omega) = X_1(\omega) + X_2(\omega)$$

example: convolution

- Determine the Fourier transform of the sequence

$$x_1(n) = x_2(n) = \{1, 1, 1\}$$

Solution

Since $x(n)$ is real and even,

$$X_1(\omega) = X_2(\omega) = 1 + 2\cos\omega$$

Using the convolution property of the Fourier transform

$$\begin{aligned} X(\omega) &= X_1(\omega)X_2(\omega) = (1 + 2\cos\omega)^2 \\ &= 3 + 4\cos\omega + 2\cos 2\omega \\ &= 3 + 2(e^{j\omega} + e^{-j\omega}) + (e^{j2\omega} + e^{-j2\omega}) \end{aligned}$$

$$x(n) = \{1, 2, 3, 2, 1\}$$

Properties of the Fourier Transform for Discrete-time signals

Wiener-Khintchine theorem

For $x(n)$, a real signal

$$r_{xx}(l) \stackrel{F}{\longleftrightarrow} S_{xx}(\omega)$$

- Energy spectral density of an energy signal is the Fourier transform of the signal autocorrelation sequence.
- Recall, $r_{x_1x_2}(m) \stackrel{F}{\longleftrightarrow} S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$

Properties of the Fourier Transform for Discrete-time signals

Parseval's theorem

If

$$x_1(n) \stackrel{F}{\longleftrightarrow} X_1(\omega)$$

And

$$x_2(n) \stackrel{F}{\longleftrightarrow} X_2(\omega)$$

Then

$$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega) X_2^*(\omega) d\omega$$

Properties of the Fourier Transform, Parseval's theorem

- Proof: starting with the right side

$$\frac{1}{2\pi}\int_{2\pi}\left[\sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n}\right] X_2^*(\omega)d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)\frac{1}{2\pi}\int_{2\pi} X_2^*(\omega)e^{-j\omega n}d\omega = \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$$

- For $x(n) = x_1(n) = x_2(n)$, Parseval's reduces to:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\omega)|^2 d\omega$$

Properties of the Fourier Transform, Parseval's theorem

- Can use Parsevals' theorem to find:

$$E_x = r_{xx}(0) = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\omega|^2 d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}(\omega) d\omega$$

- Example: refer back to 4.4.4 $E_x = 3$, and $r_{xx}(0) = 3$

Differentiation in the frequency domain

$$x(n) \overset{F}{\longleftrightarrow} X(\omega)$$
$$nx(n) \overset{F}{\longleftrightarrow} j\frac{dX(\omega)}{d\omega}$$

- Proof:

$$\begin{aligned}
\frac{dX(\omega)}{d\omega} &= \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right] \\
&= \sum_{n=-\infty}^{\infty} x(n)\frac{d}{d\omega}e^{-j\omega n} \\
&= -j\sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n}
\end{aligned}$$