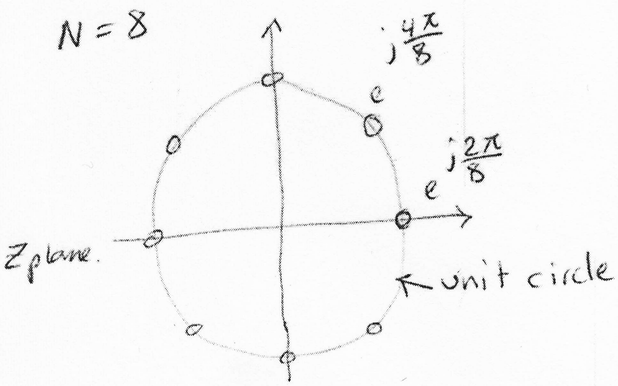
problem 8.1

Show that each of the numbers $e^{j\left(\frac{2\pi}{N}\right)k}$, $0 \le k \le N-1$ corresponds to an Nth root of unity. Plot these numbers as phasors in the complex plane and illustrate, by means of this figure, the orthogonality property.

$$\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)kn} e^{-j\left(\frac{2\pi}{N}\right)\ell n} = \begin{cases} N, & \text{if } \operatorname{mod}(k-\ell, N) = 0 \\ 0, & \text{o/w} \end{cases}$$

let $\left[e^{j\left(\frac{2\pi}{N}\right)k}\right]^N = e^{j2\pi k} = 1 \Rightarrow x^N = 1$ i.e. $e^{j\frac{2\pi k}{N}}$ is the N^{th} root of unity.

$N = 8$



now consider:

$$\sum_{n=0}^{N-1} e^{j\left(\frac{2\pi}{N}\right)kn} e^{-j\left(\frac{2\pi}{N}\right)\ell n}$$

→ if $k \neq \ell$, the sum represents the N equally spaced roots on the unit circle, the roots sum to zero.

→ if $k = \ell$, sum $\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-\ell)n} = N$

problem 8.3

let $x(n)$ be a real-valued N-point ($N = 2^{\nu}$) sequence. Develop a method to compute an N-point DFT ~~$\not{X}$~~ $X'(k)$, which contains only the odd harmonics [i.e. $X'(k) = 0$ if k is even] by using only a real N/2-point DFT.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad , \quad 0 \leq k \leq N-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{r=0}^{\frac{N}{2}-1} x(r+\tfrac{N}{2}) W_N^{(r+\frac{N}{2})k}$$

so to keep odd harmonics.

let $X'(k') = X(2k+1) \quad , \quad 0 \leq k' \leq \frac{N}{2}-1$

$$\Rightarrow X'(k') = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{(2k'+1)n} + \sum_{r=0}^{\frac{N}{2}-1} x(r+\tfrac{N}{2}) W_N^{(r+\frac{N}{2})(2k'+1)}$$

knowing $W_N^{2k'n} = W_{\frac{N}{2}}^{k'n}$, $W_N^{N} = 1$

$$\Rightarrow X'(k') = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) W_N^{n} W_{\frac{N}{2}}^{k'n} + x(n+\tfrac{N}{2}) W_{\frac{N}{2}}^{k'n} W_N^{n} W_N^{\frac{N}{2}} \right]$$

$$W_N^{N/2} = e^{-j\frac{2\pi}{N}\cdot\frac{N}{2}} = -1$$

$$\boxed{X'(k') = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x(n+\tfrac{N}{2}) \right] W_N^{n} W_{\frac{N}{2}}^{k'n}}$$

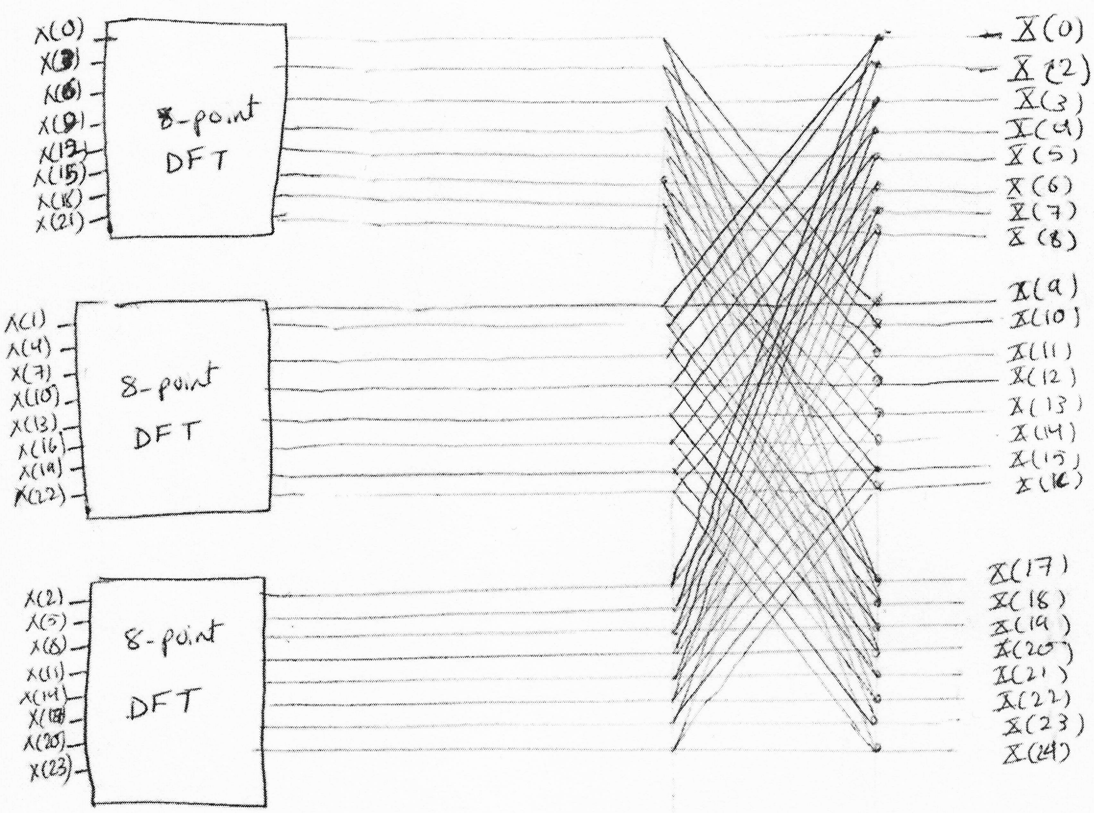
[problem 8.4] A designer has available a number of eight-po FFT chips. Show explicitly how he would interconnect three such chips in order to compute a 24-point DFT

$$X(k) = \sum_{n=0,3,6,\ldots}^{21} x(n) W_N^{kn} + \sum_{n=1,4,7,\ldots}^{22} x(n) W_N^{kn} + \sum_{n=2,5,8,\ldots}^{23} x(n) W_N^{kn}$$

$$= \sum_{r=0}^{7} x(3r) W_{\frac{N}{3}}^{kr} + \sum_{r=0}^{7} x(3r+1) W_N^{(3r+1)k} + \sum_{r=0}^{7} x(3r+2) W_N^{(3r+2)k}$$

$$= \sum_{r=0}^{7} x(3r) W_{\frac{N}{3}}^{kr} + \sum_{r=0}^{7} x(3r+1) W_{\frac{N}{3}}^{kr} W_N^{k} + \sum_{r=0}^{7} x(3r+2) W_{\frac{N}{3}}^{kr} W_N^{2k}$$

$$= F_1(k) + F_2(k) W_N^{k} + F_3(k) W_N^{2k}$$



$e^{-j\frac{2\pi}{4}\cdot\frac{N}{2}}$

$e^{-j\pi}$

problem 8.8 Compute the eight-point DFT of the sequence

$$x(n) = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & \text{ow} \end{cases}$$

by using the decimation-in-frequency FFT algorithm described in text.

following fig. 8.1.11

1^{st} stage $\rightarrow$ $\{2, 2, 2, 2, 0, 0, 0, 0\}$

2^{nd} stage $\rightarrow$ $\{4, 4, 0, 0, 0, 0, 0, 0\}$

3^{rd} stage $\rightarrow$ $\{8, 0, 0, 0, 0, 0, 0, 0\}$

reodering $\rightarrow$ remains $\{8, 0, 0, 0, 0, 0, 0, 0\}$

problem 8.1 Compute the 8-point DFT of the sequence

$$x(n) = \left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right\}$$

using the in-place ~~2~~ radix-2 decimation-in-time & radix-2 decimation in-frequency algorithms. Follow exactly the corresponding signal flow graphs and keep track of all the intermediate quantities by putting them on the diagrams.

Figure 8.1.6 Decimation-in-time FFT

→ 1^{st} stage output → $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right\}$

→ 2^{nd} stage output → $\Big\{ \underbrace{\left(\frac{1}{2}+\frac{1}{2}\cdot W_8^0\right)}_{1}, \frac{1}{2}(1+W_8^2), \underbrace{\frac{1}{2}-\frac{1}{2}W_8^0}_{0}, \frac{1}{2}(1-W_8^2), 1, \frac{1}{2}(1+W_8^2), 0, \frac{1}{2}(1-W_8^2) \Big\}$

→ 3^{rd} stage output → $\Big\{ 2, 0, 0, 0, \frac{1}{2}(1+W_8^1+W_8^2+W_8^3), \frac{1}{2}(1-W_8^1+W_8^2-W_8^3), \frac{1}{2}(1-W_8^2+W_8^3-W_8^5), \frac{1}{2}(1-W_8^2-W_8^3+W_8^5) \Big\}$

following Figure 8.1.11 for Decimation-in-frequency FFT

→ 1^{st} stage: $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}W_8^1, \frac{1}{2}W_8^2, \frac{1}{2}W_8^3\right\}$

→ 2^{nd} stage: $\left\{1, 1, 0, 0, \frac{1}{2}+\frac{1}{2}W_8^2, \frac{1}{2}W_8^1+\frac{1}{2}W_8^3, \frac{1}{2}-\frac{1}{2}W_8^2, \frac{1}{2}W_8^3-\frac{1}{2}W_8^5\right\}$

→ 3^{rd} stage: $\Big\{ 2, 0, 0, 0, \frac{1}{2}(1+W_8^1+W_8^2+W_8^3), \frac{1}{2}(1-W_8^1+W_8^2-W_8^3), \frac{1}{2}(1-W_8^2+W_8^3-W_8^5), \frac{1}{2}(1-W_8^2-W_8^3+W_8^5) \Big\}$

$\boxed{8.13}$ Consider the eight-point decimation-in-time algorithm (DIT) flow graph in 8.1.6.

(a) What is the gain of the "signal path" that goes from $x(7)$ to $X(2)$?

$\Rightarrow$ "gain" $= W_8^0 W_8^0 (-1) W_8^2 = -W_8^2 = -e^{-j\frac{2\pi \cdot 2}{8}} = j$

(b) How many paths lead from the input to a given output sample? Is this true for every output sample?

$\Rightarrow$ Given a certain output sample, there is one path from every input leading to it. This is true for every output.

(c) Compute $X(3)$ using the operations dictated by this flow graph.

$\Rightarrow X(3) = x(0) + W_8^3 x(1) - W_8^2 x(2) + W_8^2 W_8^3 x(3) - W_8^0 x(4) - W_8^0 W_8^3 x(5)$

$+ W_8^0 W_8^2 x(6) + W_8^0 W_8^2 W_8^3 x(7)$.

$\boxed{8.16}$ Show that the product of two complex numbers $(a+jb)$ and $(c+jd)$ can be performed with three real ~~number~~ multiplications and five additions using the algorithm

$$x_R = (a-b)d + (c-d)a$$

$$x_I = (a-b)d + (c+d)b.$$

Where $x = x_R + jx_I = (a+jb)(c+jd)$.

$\Rightarrow$ From the algorithm, $(a-b)d$ is common for computation of x_R and x_I. We have $(a-b)d$, $(c-d)a$, and $(c+d)b$, each of which has 1 add and 1 multiplication. We need one more add for x_R and x_I, respectively.

$\Rightarrow 3\times1+2 = 5$ add and $3\times1 = 3$ multiplication.

$\boxed{8.19}$ Let $X(k)$ be the N-point DFT of the sequence $x(n), 0 \le n \le N-1$. What is the N-point DFT of the sequence $s(n) = X(n), 0 \le n \le N-1$?

$\Rightarrow$ Known $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$. Let $F(t)$ be the DFT of the sequence $s(n) = X(k)$.

$F(t) = \sum_{k=0}^{N-1} X(k) W_N^{tk} = \sum_{k=0}^{N-1} \left(\sum_{n=0}^{N-1} x(n) W_N^{kn}\right) W_N^{tk} = \sum_{n=0}^{N-1} x(n) \left[\sum_{k=0}^{N-1} W_N^{kn} W_N^{tk}\right]$

$= \sum_{n=0}^{N-1} x(n)\, \delta((n+t))_N = \sum_{n=0}^{N-1} x(n)\, \delta(N-1-n-t) \quad t = 0, 1, \cdots, N-1$

$= \{x(N-1), x(N-2), \cdots, x(1), x(0)\}$.

$\boxed{8.20}$ Let $X(k)$ be the N-point DFT of the sequence $x(n)$, $0 \le n \le N-1$. We define a 2N-point sequence $y(n)$ as $y(n) = \begin{cases} x(\frac{n}{2}), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$. Express the 2N-point DFT of $y(n)$ in terms of $X(k)$.

$$\Rightarrow Y(k) = \sum_{n=0}^{2N-1} y(n) W_{2N}^{kn} = \sum_{n=0,\, n \text{ even}}^{2N-1} y(n) W_{2N}^{kn} = \sum_{m=0}^{N-1} y(2m) W_{2N}^{2km}$$

$$= \sum_{m=0}^{N-1} y(2m) W_N^{km} = \sum_{m=0}^{N-1} x(m) W_N^{km} = X(k), \quad k \in [0, N-1]$$

$$= X(k-N), \quad k \in [N, 2N-1].$$

$\boxed{8.35}$. The basic butterfly in the radix-2 decimation-in-time FFT algorithm is

$$X_{n+1}(k) = X_n(k) + W_N^m X_n(l).$$

$$X_{n+1}(l) = X_n(k) - W_N^m X_n(l).$$

(a) If we require that $|X_n(k)| < \frac{1}{2}$, and $|X_n(l)| < \frac{1}{2}$, show that

$$|\text{Re}[X_{n+1}(k)]| < 1, \quad |\text{Re}[X_{n+1}(l)]| < 1$$

$$|\text{Im}[X_{n+1}(k)]| < 1, \quad |\text{Im}[X_{n+1}(l)]| < 1.$$

$\Rightarrow$ Thus overflow does not occur.

$$\Rightarrow \text{Re}[X_{n+1}(k)] = \frac{1}{2} X_{n+1}(k) + \frac{1}{2} X^*_{n+1}(k)$$

$$= \frac{1}{2} X_n(k) + \frac{1}{2} W_N^m X_n(l) + \frac{1}{2} X_n^*(k) - \frac{1}{2} W_N^{-m} X_n^*(l)$$

$$= \text{Re}[X_n(k)] + \text{Re}[W_N^m X_n(l)]$$

Since $\left\{ \begin{array}{l} |X_n(k)| < \frac{1}{2} \\ |X_n(k)| < \frac{1}{2} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} |\text{Re}[X_n(k)]| < \frac{1}{2} \\ |\text{Re}[X_l(k) W_N^m]| < \frac{1}{2} \end{array} \right.$

So $|\text{Re}[X_{n+1}(k)]| \le |\text{Re}[X_n(k)]| + |\text{Re}[W_N^m X_n(l)]| < 1$

The other inequalities are verified similarly.

(b) Prove that $\max[|X_{n+1}(k)|, |X_{n+1}(l)|] \ge \max[|X_n(k)|, |X_n(l)|]$.

$$\max[|X_{n+1}(k)|, |X_{n+1}(l)|] \le 2\max[|X_n(k)|, |X_n(l)|].$$

$$\Rightarrow X_{n+1}(k) = \{\text{Re}[X_n(k)] + j\,\text{Im}[X_n(k)]\} + \{\cos(\tfrac{2\pi}{N}m) - j\sin(\tfrac{2\pi}{N}m)\}\{\text{Re}[X_n(l)] + j\,\text{Im}[X_n(l)]\}$$

$$= Re[X_n(k)] + \cos(\tfrac{2\pi}{N}m) Re[X_n(l)] + \sin(\tfrac{2\pi}{N}m) Im[X_n(l)] + j\{Im[X_n(k)] + \cos(\tfrac{2\pi}{N}m) Im[X_n(l)]$$

$$+ \sin(\tfrac{2\pi}{N}m) \cancel{Re}[X_n(l)]\}$$

Therefore, $|X_{n+1}(k)| = |X_n(k)| + |X_n(l)| + A$,

where $A = 2\cos(\frac{2\pi}{N}m)\{Re[X_n(k)]Re[X_n(l)] + Im[X_n(k)]Im(X_n(l))\}$
$+ 2\sin(\frac{2\pi}{N}m)\{Re[X_n(k)]Im[X_n(l)] - Im[X_n(k)]Re[X_n(l)]\}$

also $|X_{n+1}(l)|^2 = |X_n(k)|^2 + |X_n(l)|^2 - A$

Therefore, if $A \geq 0$

$$\max[|X_{n+1}(k)|, |X_{n+1}(l)|] = |X_{n+1}(k)|$$
$$= \{|X_n(k)|^2 + |X_n(l)|^2 + A\}^{\frac{1}{2}}$$
$$\geq \max[|X_n(k)|, |X_n(l)|].$$

We can prove that the inequality hold if $A < 0$.

Also, from the pair of equation for computing the butterfly outputs, we have

$$2X_n(k) = X_{n+1}(k) + X_{n+1}(l)$$
$$2X_n(l) = W_N^{-m} X_{n+1}(k) - W_N^{-m} X_{n+1}(l).$$

Based above two equations, by a similar method to that employed above, it can be shown that

$$2\max[|X_n(k)|, |X_n(l)|] \geq \max[|X_{n+1}(k)|, |X_{n+1}(l)|].$$

problem 8.25 Develop a radix-3 decimation-in-time FFT algorithm for $N = 3^{\nu}$, and draw the corresponding flow graph for $N = 9$. What is the number of required complex multiplications? can the operations be performed in place?

$$X(k) = \sum_{n=0}^{8} x(n) W_9^{nk} = \sum_{n=0,3,6} x(n) W_9^{nk} + \sum_{n=1,4,7} x(n) W_9^{nk} + \sum_{n=2,5,8} x(n) W_9^{nk}$$

$$= \sum_{m=0}^{\frac{N}{3}-1=2} x(3m) W_9^{(3m)k} + \sum_{m=0}^{2} x(3m+1) W_9^{(3m+1)k} + \sum_{m=0}^{2} x(3m+2) W_9^{(3m+2)k}$$

$$= \sum_{m=0}^{2} x(3m) W_{\frac{9}{3}}^{mk} + W_9^{k} \sum_{m=0}^{2} x(3m+1) W_3^{km} + W_9^{2k} \sum_{m=0}^{2} x(3m+2) W_3^{mk}$$

$$= F_1(k) + W_9^{k} F_2(k) + W_9^{2k} F_3(k), \quad 0 \le k \le N-1$$

✻ exploiting the periodicity of $F_1(k), F_2(k), F_3(k)$, for $0 \le k \le \frac{N}{3}-1$

$$X(k) = F_1(k) + W_9^{k} F_2(k) + W_9^{2k} F_3(k)$$

$$X(k+\tfrac{N}{3}) = F_1(k) + W_3^{1} W_9^{k} F_2(k) + W_3^{2} W_9^{2k} F_3(k)$$

$$X(k+\tfrac{2N}{3}) = F_1(k) + W_3^{2} W_9^{k} F_2(k) + W_3^{4} W_9^{2k} F_3(k)$$

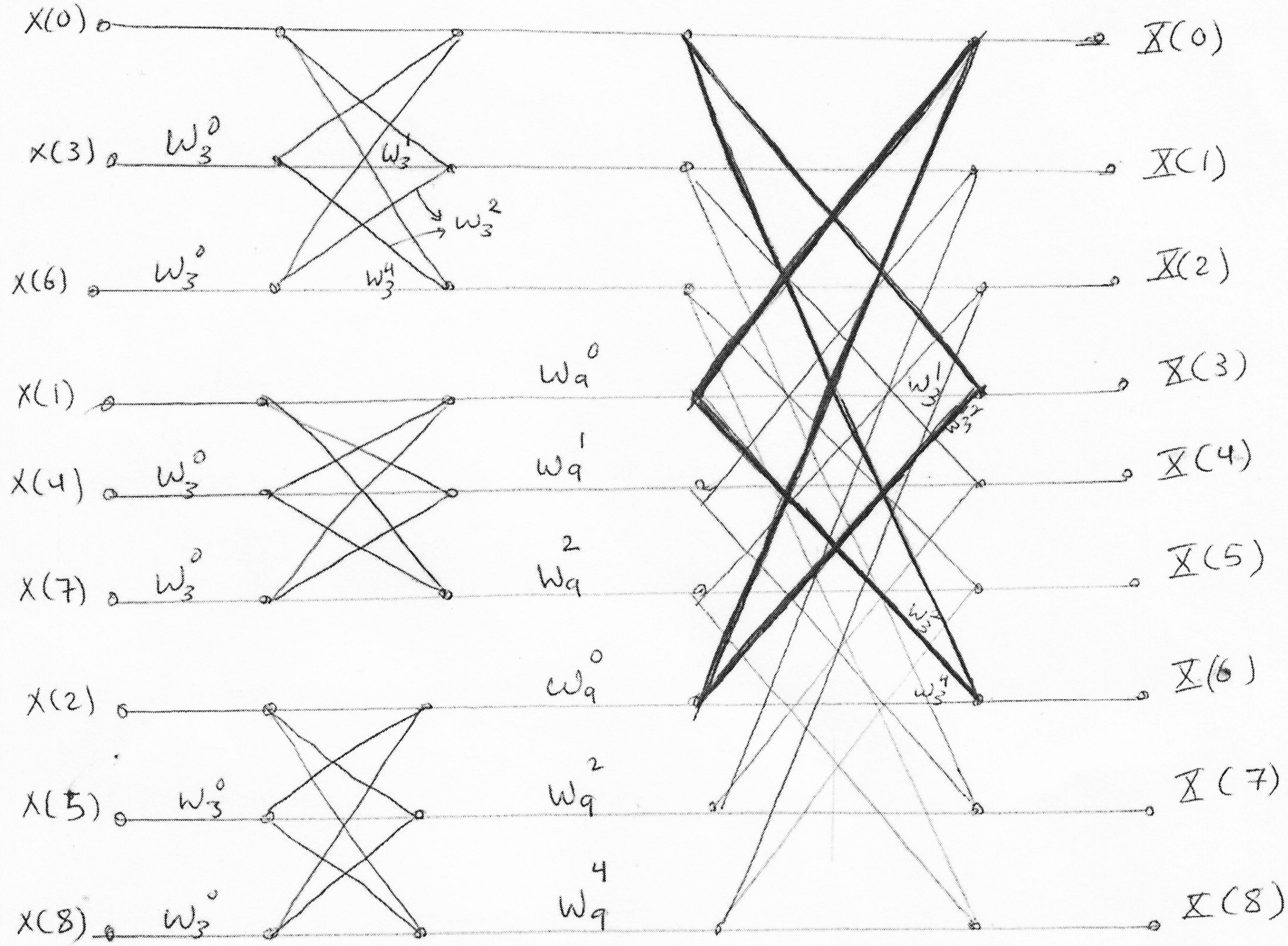
✻ we have $\log_3 N = 2$ stages.

# of complex multiplications:

✻ in each stage → 6 complex multiplication by twiddle factors + 12 multiplications inside the butterfly's.

⇒ total # of complex multiplications = 36.

there are 8 - $W^0 = 1$ ⇒ # of multiplications = 28


x(0)
x(3)
x(6)
x(1)
x(4)
x(7)
x(2)
x(5)
x(8)
W_3^0
W_3^1
W_3^2
W_3^4
W_9^0
W_9^1
W_9^2
W_9^4
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)
X(8)