3.1

(a) $x(n) = \{3, 0, 0, 0, 0, \underset{\uparrow}{6}, 1, -4\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= 3 \cdot z^{-(-5)} + 6 \cdot z^0 + 1 \cdot z^{-1} + (-4) \cdot z^{-2}$$

$$= 3z^5 + 6 + z^{-1} - 4z^{-2}$$

ROC: $0 < |z| < \infty$

(b) $x(n) = \begin{cases} (\frac{1}{2})^n, & n \geq 5 \\ 0, & n \leq 4 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=5}^{\infty} (\tfrac{1}{2})^n z^{-n}$$

$$= \sum_{n=5}^{\infty} (\tfrac{1}{2z})^n$$

Thus the ROC: $\left|\frac{1}{2z}\right| < 1 \Rightarrow |z| > \frac{1}{2}$

Then $X(z) = \dfrac{(\frac{1}{2z})^5}{1 - \frac{1}{2z}} = (\frac{1}{32}) \dfrac{z^{-5}}{1 - \frac{1}{2}z^{-1}}$

3.3.

(a) $$x_1(n) = \begin{cases} (\frac{1}{3})^n & n \geq 0 \\ (\frac{1}{2})^{-n} & n < 0 \end{cases}$$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{3})^n z^{-n} + \sum_{n=-\infty}^{-1} (\tfrac{1}{2})^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{3z})^n + \sum_{m=1}^{\infty} (\tfrac{1}{2})^m z^m \Leftarrow \text{Let } m=-n.$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{3z})^n + \sum_{m=1}^{\infty} (\tfrac{z}{2})^m$$

Thus ROC: $|\frac{1}{3z}| < 1$ & $|\frac{z}{2}| < 1 \Rightarrow \frac{1}{3} < |z| < 2$

$$X_1(z) = \frac{1}{1-\frac{1}{3z}} + \frac{\frac{z}{2}}{1-\frac{z}{2}} = \frac{\frac{5}{6}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)}$$

(b) $$x_2(n) = \begin{cases} (\frac{1}{3})^n - 2^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} [(\tfrac{1}{3})^n - 2^n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{3})^n z^{-n} - \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (\tfrac{1}{3z})^n - \sum_{n=0}^{\infty} (\tfrac{2}{z})^n$$

Thus ROC: $|\frac{1}{3z}| < 1$ & $|\frac{2}{z}| < 1 \Rightarrow 2 < |z|$

$$X_2(z) = \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$$

3.3. P3

(c) $x_3(n) = x_1(n+4)$

Thus $X_3(z) = X_1(z) z^4 \Leftarrow$ Time Shifting

Thus ROC of $X_3(z)$ is the same as that of $X_1(z)$, which is $\frac{1}{3} < |z| < 2$.

$$X_3(z) = X_1(z) z^4 = \frac{\frac{5}{6} z^4}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)}$$

(d) $x_4(n) = x_1(-n)$

Thus $X_4(z) = X_1(z^{-1}) \Leftarrow$ Time - Reversal

$$= \frac{\frac{5}{6}}{(1-\frac{1}{3}z)(1-\frac{1}{2}z^{-1})}$$

ROC : $\frac{1}{2} < |z| < 3$

P4

3.5 Let the two-sided sequence $x(n)$ has the z-transform with ROC as $r_1<|z|<r_2$, where $r_1, r_2 \geq 0$

Let the right-sided sequence $x_r(n) = \begin{cases} x(n) & \text{if } n \geq n_0 \\ 0 & \text{otherwise} \end{cases}$

the left-sided sequence $x_l(n) = \begin{cases} x(n) & \text{if } n \leq n_0 \\ 0 & \text{otherwise} \end{cases}$

the finite-duration two-sided sequence $x_f(n) = \begin{cases} x(n) & \text{if } n_0 \leq n \leq n_1 \\ 0 & \text{otherwise} \end{cases}$

where $n_0 < 0 < n_1$.

(a) If $n_0 \geq 0$, then $X_r(z) = \sum_{n=-\infty}^{\infty} x_r(n) z^{-n} = \sum_{n=n_0}^{\infty} x(n) z^{-n}$

Thus ROC is $|z| > r_1$

If $n_0 < n$ then $X_r(z) = \sum_{n=-\infty}^{\infty} x_r(n) z^{-n} = \sum_{n=n_0}^{\infty} x(n) z^{-n}$

$$= \underbrace{\sum_{n=0}^{\infty} x(n) z^{-n}}_{①} + \underbrace{\sum_{n=n_0}^{-1} x(n) z^{-n}}_{②}$$

① converges for $|z| > r_1$, and ② converges for $|z| < \infty$

Therefore ROC is $r_1 < |z| < \infty$

(b) $X_l(z) = \sum_{n=-\infty}^{\infty} x_l(n) z^{-n} = \sum_{n=-\infty}^{n_0} x(n) z^{-n}$

If $n_0 \leq 0$, then ROC is $|z| < r_2$

If $n_0 > 0$, then $X_l(z) = \underbrace{\sum_{n=-\infty}^{0} x(n) z^{-n}}_{①} + \underbrace{\sum_{n=1}^{n_0} x(n) z^{-n}}_{②}$

① converges for $|z| < r_2$, and ② converges for $|z| > 0$.

Thus ROC: $0 < |z| < r_2$.

(c) $X_f(z) = \sum_{n=-\infty}^{\infty} x_f(n) z^{-n} = \sum_{n=n_0}^{n_1} x(n) z^{-n} = \underbrace{\sum_{n=n_0}^{0} x(n) z^{-n}}_{①} + \underbrace{\sum_{n=1}^{n_1} x(n) z^{-n}}_{②}$

① converges for $|z| < \infty$, and ② converges for $|z| > 0$

Thus ROC is: $0 < |z| < \infty$.

(5)

3.6. $y(n) = \sum_{k=-\infty}^{n} x(k)$

Thus $y(n) - y(n-1) = \sum_{k=-\infty}^{n} x(k) - \sum_{k=-\infty}^{n-1} x(k) = x(n)$.

Based on "time shifting", let $y(n) \xrightarrow{z} Y(z)$, we have $y(n-1) \xrightarrow{z} z^{-1} Y(z)$

Thus $y(n) - y(n-1) \xrightarrow{z} Y(z) - z^{-1} Y(z)$.

Since $y(n) - y(n-1) = x(n)$ and $x(n) \xrightarrow{z} X(z)$,

we have $Y(z) - z^{-1} Y(z) = X(z)$.

Therefore $Y(z) = \dfrac{X(z)}{1 - z^{-1}}$

$\boxed{3.7}$ Compute the convolution of the following signals by means of the z-transform.

$$x_1(n) = \begin{cases} (\frac{1}{3})^n, & n \geq 0 \\ (\frac{1}{2})^{-n}, & n < 0 \end{cases}, \quad x_2(n) = (\frac{1}{2})^n u(n).$$

$$\Rightarrow X_1(z) = \sum_{n=0}^{\infty} (\frac{1}{3})^n z^{-n} + \sum_{n=-\infty}^{-1} (\frac{1}{2})^{-n} z^{-n}$$

$$= \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z} - 1$$

$$= \frac{5/6}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)}, \quad \frac{1}{3} < |z| < 2$$

$$X_2(z) = \sum_{n=0}^{\infty} (\frac{1}{2})^n z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}$$

Let $y(n) = x_1(n) * x_2(n)$,

$$Y(z) = X_1(z) X_2(z)$$

$$= \frac{5/6}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z)(1-\frac{1}{2}z^{-1})}$$

$$= \frac{-2}{1-\frac{1}{3}z^{-1}} + \frac{10/3}{1-\frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1-\frac{1}{2}z}, \quad \frac{1}{2} < |z| < 2$$

$$\Rightarrow y(n) = \begin{cases} -2(\frac{1}{3})^n + \frac{10}{3}(\frac{1}{2})^n, & n \geq 0 \\ \frac{4}{3}(\frac{1}{2})^{-n}, & n < 0 \end{cases}$$

$\boxed{3.9}$ The z-transform $X(z)$ of a real signal $x(n)$ includes a pair of complex-conjugate zeros and a pair of complex-conjugate poles. What happens to these pairs if we multiply $x(n)$ by $e^{j\omega_0 n}$? (Hint: Use the scaling theorem in the z-domain).

$\Rightarrow$ Let $y(n) = x(n) e^{j\omega_0 n}$.

From the scaling theorem, we have $Y(z) = X(e^{-j\omega_0} z)$. Thus, the poles and zeros are phase rotated by an angle $\omega_0$.