3.12. Note: there is a typo in textbook.

$$X(z) = \frac{1}{(1-2z^{-1})(1-z^{-1})^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z-2)(z-1)^2} = \frac{A_1}{(z-2)} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2}$$

$$A_1 = \frac{z^2}{(z-1)^2}\Big|_{z=2} = 4$$

$$A_3 = \frac{z^2}{(z-2)}\Big|_{z=1} = -1$$

$$A_2 = \frac{d}{dz}\left[\frac{z^2}{(z-2)}\right]\Big|_{z=1} = -3$$

$$X(z) = \frac{4}{1-2z^{-1}} - \frac{3}{1-z^{-1}} - \frac{z^{-1}}{(1-z^{-1})^2}$$

Based on $a^n u(n) \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}$ and $na^n u(n) \overset{z}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}$

we have $x(n) = [4(2)^n - 3 - n]\, u(n)$.

3.13.

(a) $x_1(n) = \begin{cases} x(\frac{n}{2}) & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} = \sum_{n=\infty}^{\infty} x(\tfrac{n}{2}) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-2k} \Leftarrow k = \frac{n}{2}$$

$$= \sum_{k=-\infty}^{\infty} x(k) (z^2)^{-k}$$

$$= X(z^2)$$

(b) $x_2(n) = x(2n)$

$$X_2(n) = \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(2n) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-\frac{k}{2}} \Leftarrow k = 2n., \text{ thus } k \text{ is even}$$

$$= \sum_{k=-\infty}^{\infty} \left[\frac{x(k) + (-1)^k x(k)}{2}\right] z^{-\frac{k}{2}}, \Leftarrow k \text{ is any integer}$$

$$= \frac{1}{2}\sum_{k=\infty}^{\infty} x(k) (z^{\frac{1}{2}})^{-k} + \frac{1}{2}\sum_{k=-\infty}^{\infty} x(k) (-z^{-\frac{1}{2}})^{-k}$$

$$= \frac{1}{2}X(z^{\frac{1}{2}}) + \frac{1}{2}X(-z^{\frac{1}{2}})$$

3.14.

(a) $X(z) = X_1(z)(1+3z^{-1})$

where $X_1(z) = \frac{1}{1+3z^{-1}+2z^{-2}} = \frac{1}{(1+z^{-1})(1+2z^{-1})}$

$$X_1(z) = \frac{1}{(1+z^{-1})(1+2z^{-1})} = \frac{z^2}{(z+1)(z+2)}$$

$$\frac{X_1(z)}{z} = \frac{z}{(z+1)(z+2)} = \frac{A_1}{z+1} + \frac{A_2}{z+2}$$

$$A_1 = \frac{z}{z+2}\Big|_{z=-1} = -1, \quad A_2 = \frac{z}{z+1}\Big|_{z=-2} = 2$$

Thus $X_1(z) = \frac{-1}{1+z^{-1}} + \frac{2}{1+2z^{-1}}$

$x_1(n) = -(-1)^n u(n) + 2(-2)^n u(n)$

Since $X(z) = X_1(z)(1+3z^{-1})$

we have $x(n) = x_1(n) + 3x_1(n-1) \Leftarrow \begin{cases} \text{linearity} \\ \text{time-shifting} \end{cases}$

Thus $x(n) = -(-1)^n u(n) + 2(-2)^n u(n) - 3(-1)^{n-1} u(n-1)$
$+ 6(-2)^{n-1} u(n-1)$

$= -(-1)^n[u(n) - u(n-1)] + 2(-1)^n u(n-1)$
$+ 2(-2)^n[u(n) - u(n-1)] - (-2)^n u(n-1)$

$= -(-1)^n \delta(n) + 2(-2)^n \delta(n)$
$+ 2(-1)^n u(n-1) - (-2)^n u(n-1)$

$= 2(-1)^n u(n) - (-2)^n u(n)$

Alternative method:

$$X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} = \frac{2}{1+z^{-1}} + \frac{-1}{1+2z^{-1}}$$

$x(n) = [2(-1)^n - (-2)^n] u(n)$

3.14

(i). $X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}$

$= (1-\frac{1}{2}z^{-1})X_1(z)$, where $X_1(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}$

$x(n) = x_1(n) - \frac{1}{2}x_1(n-1) \Leftarrow \begin{cases} \text{linearity} \\ \text{time-shifting} \end{cases}$

Since $X_1(z) = \dfrac{1}{1+\frac{1}{2}z^{-1}}$ and $x_1(n)$ is causal

we have $x_1(n) = (-\frac{1}{2})^n u(n) \Leftarrow \left[a^n u(n) \overset{z}{\longleftrightarrow} \dfrac{1}{1-az^{-1}}\right]$

Thus $x(n) = (-\frac{1}{2})^n u(n) - \frac{1}{2}(-\frac{1}{2})^{n-1} u(n-1)$

$= (-\frac{1}{2})^n u(n) + (-\frac{1}{2})^n u(n-1)$

(j). $X(z) = \dfrac{1-az^{-1}}{z^{-1}-a} = (1-az^{-1})X_1(z)$, where $X_1(z) = \dfrac{1}{z^{-1}-a}$.

Thus $x(n) = x_1(n) - a x_1(n-1)$.

Since $X_1(z) = \dfrac{1}{z^{-1}-a} = \dfrac{-\frac{1}{a}}{1-\frac{1}{a}z^{-1}}$ and $x_1(n)$ is causal

we have $x_1(n) = -\frac{1}{a}(\frac{1}{a})^n u(n) = -(\frac{1}{a})^{n+1} u(n)$

Thus $x(n) = -(\frac{1}{a})^{n+1} u(n) + a(\frac{1}{a})^n u(n-1)$

$= -(\frac{1}{a})^{n+1} u(n) + (\frac{1}{a})^{n-1} u(n-1)$.

$\boxed{3.15}$

$$X(z) = \frac{5z^{-1}}{(1-2z^{-1})(3-z^{-1})}$$

$$= \frac{1}{1-2z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}}$$

For $|z|>2$, $x(n) = \left[2^n - \left(\frac{1}{3}\right)^n\right] u(n)$.

For $|z|<\frac{1}{3}$, $x(n) = \left(\frac{1}{3}\right)^n u(-n-1) - 2^n u(-n-1)$

For $\frac{1}{3}<|z|<2$, $x(n) = -\left(\frac{1}{3}\right)^n u(n) - 2^n u(-n-1)$

$\boxed{3.18}$

(a) $$\sum_{n=-\infty}^{\infty} x^*(n) z^{-n} = \left(\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n}\right)^*$$

$$= X^*(z^*)$$

(b) $$Z[Re\{x(n)\}] = Z\left[\frac{1}{2}[x(n) + x^*(n)]\right]$$

$$= \frac{1}{2} Z[x(n)] + \frac{1}{2} Z[x^*(n)]$$

$$= \frac{1}{2}[X(z) + X^*(z^*)]$$

(c) $$Z[Im\{x(n)\}] = Z\left[\frac{1}{2j}(x(n) - x^*(n))\right]$$

$$= \frac{1}{2j}\left[Z[x(n)] - Z[x^*(n)]\right]$$

$$= \frac{1}{2j}[X(z) - X^*(z^*)]$$

(d) $$X_k(z) = \sum_{\substack{n=-\infty \\ n/k \text{ integer}}}^{\infty} x\left(\frac{n}{k}\right) z^{-n} = \sum_{m=-\infty}^{\infty} x(m) z^{-mk}$$

$$= X(z^k)$$

(e) $$\sum_{n=-\infty}^{\infty} e^{j\omega_0 n} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n) \left(e^{-j\omega_0} z\right)^{-n}$$

$$= X(e^{-j\omega_0} z)$$

3.23.

$$X(z) = e^{z} + e^{\frac{1}{z}}, \qquad |z| \neq 0$$

Since $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x$

we have $X(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{(\frac{1}{z})^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}$$

$$= \sum_{n=-\infty}^{-1} \frac{z^{-n}}{n!} + \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} + 1$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{n!}\right) z^{-n} + 1$$

Thus $x(n) = \frac{1}{n!} + \delta(n)$

$\boxed{3.24}$ (a) $$X(z) = \frac{1}{1+1.5z^{-1}-0.5z^{-2}}$$

$$= \frac{0.136}{1-0.28z^{-1}} + \frac{0.864}{1+1.78z^{-1}}$$

Hence $x(n) = [0.136(0.28)^n + 0.864(-1.78)^n]\,u[n]$.

(b) $$X(z) = \frac{1}{1-0.5z^{-1}+0.6z^{-2}}$$

$$= \frac{1-0.25z^{-1}}{1-0.5z^{-1}+0.6z^{-2}} + 0.3412\frac{0.7326z^{-1}}{1-0.5z^{-1}+0.6z^{-2}}$$

Hence $x(n) = (0.7746)^n[\cos 1.24n + 0.3412\sin 1.24n]\,u(n)$.

partial check: $x(0)=1$, $x(1)=0.5016$, $x(2)=-0.3476$, $x(\infty)=0$.

From difference equation, $x(n)-0.5x(n-1)+0.6x(n-2)=\delta(n)$, we get $x(0)=1$, $x(1)=0.5$, $x(2)=-0.35$, $x(\infty)=0$.

$\boxed{3.25}$ (a) $$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}} = \frac{1}{(1-z^{-1})(1-0.5z^{-1})}$$

$$= \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

For $|z|<0.5$, $x(n) = [(0.5)^n - 2]\,u[-n-1]$

For $|z|>1$, $x(n) = [2-(0.5)^n]\,u[n]$

For $0.5<|z|<1$, $x(n) = -(0.5)^n u(n) - 2u(-n-1)$

(b) $$X(z) = \frac{1}{1-0.5z^{-1}+0.25z^{-2}} = \frac{1}{\left(1-\frac{1-\sqrt{3}j}{4}z^{-1}\right)\left(1-\frac{1+\sqrt{3}j}{4}z^{-1}\right)}$$

$$= \frac{\frac{1}{6}(3+\sqrt{3}j)}{1-\frac{1-\sqrt{3}j}{4}z^{-1}} + \frac{\frac{1}{6}(3-\sqrt{3}j)}{1-\frac{1+\sqrt{3}j}{4}z^{-1}}$$

For $|z|<\frac{1}{2}$, $x(n) = \left[\frac{1}{6}(3+\sqrt{3}j)\left(\frac{1-\sqrt{3}j}{4}\right)^n + \frac{1}{6}(3-\sqrt{3}j)\left(\frac{1+\sqrt{3}j}{4}\right)^n\right]u(-n-1)$

For $|z|>\frac{1}{2}$, $x(n) = \frac{1}{6}\left[(3+\sqrt{3}j)\left(\frac{1-\sqrt{3}j}{4}\right)^n + (3-\sqrt{3}j)\left(\frac{1+\sqrt{3}j}{4}\right)^n\right]u(n)$.

3.28

(1) Conjugation property: $x^*(n) \overset{Z}{\longleftrightarrow} X^*(z^*)$

Let $y(n)=x^*(n)$.

Then $Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$

$$= \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[x(n)(z^{-n})^*\right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)(z^*)^{-n}\right]^*$$

$$= X^*(z^*)$$

(2) Parseval's relation: $\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v) X_2^*\left(\frac{1}{v^*}\right) v^{-1} dv$

Since $x(n) = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$ (Eq.(3.1.16) in P157 Textbook)

we have $\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi j}\oint_C X_1(z) z^{n-1} dz \, x_2^*(n)\right]$

$x_2^*(n)$ is constant with respect to the integration. $\Rightarrow$ $= \frac{1}{2\pi j}\oint_C X_1(z)\left[\sum_{n=-\infty}^{\infty} x_2^*(n)\left(\frac{1}{z}\right)^{-n}\right] z^{n-1} dz$

$x^*(n) \overset{Z}{\longleftrightarrow} X^*(z^*)$ $\Rightarrow$ $= \frac{1}{2\pi j}\oint_C X_1(z) X_2^*\left(\frac{1}{z^*}\right) dz$

3.32.

(a) $y(n) = 0.2y(n-1) + x(n) - 0.3x(n-1) + 0.02x(n-2)$.

$$Y(Z) = 0.2Z^{-1}Y(Z) + X(Z) - 0.3Z^{-1}X(Z) + 0.02Z^{-2}X(Z)$$

$$(1-0.2Z^{-1})Y(Z) = (1-0.3Z^{-1}+0.02Z^{-2})X(Z)$$

$$H_1(Z) = \frac{Y(Z)}{X(Z)} = \frac{1-0.3Z^{-1}+0.02Z^{-2}}{1-0.2Z^{-1}}$$

$$= \frac{(1-0.1Z^{-1})(1-0.2Z^{-1})}{1-0.2Z^{-1}}$$

$$= 1-0.1Z^{-1}$$

(b) $y(n) = x(n) - 0.1x(n-1)$

$$Y(Z) = X(Z) - 0.1Z^{-1}X(Z) = (1-0.1Z^{-1})X(Z)$$

$$H_2(Z) = \frac{Y(Z)}{X(Z)} = 1-0.1Z^{-1}$$

Since $H_1(Z) = H_2(Z)$, the systems in (1) and (2) are equivalent.

$\boxed{3.36}$

$$H(z) = \frac{1-2z^{-1}+2z^{-2}-z^{-3}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})}, \quad \frac{1}{2}<|z|<1$$

$$= \frac{\cancel{(1-z^{-1})}(1-z^{-1}+z^{-2})}{\cancel{(1-z^{-1})}(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})} = \frac{1-z^{-1}+z^{-2}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})}$$

(a) Zeros: $\frac{1\pm j\sqrt{3}}{2}$.

poles: $\frac{1}{2}, \frac{1}{5}$

(b) $H(z) = 1 + \left[\frac{5/2}{1-\frac{1}{2}z^{-1}} + \frac{-2.8}{1-\frac{1}{5}z^{-1}}\right]z^{-1}$

$\Rightarrow h(n) = \delta(n) + \left[5\left(\frac{1}{2}\right)^n - 14\left(\frac{1}{5}\right)^n\right]u[n]$.

$\boxed{3.37}$

$y(n) = 0.7y(n-1) - 0.12y(n-2) + x(n-1) + x(n-2)$

$Y(z) = 0.7Y(z)z^{-1} - 0.12Y(z)z^{-2} + X(z)z^{-1} + X(z)z^{-2}$

$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}+z^{-2}}{1-0.7z^{-1}-0.12z^{-2}}$

$x(n) = nu(n)$

$X(z) = \frac{z^{-1}}{(1-z^{-1})^2}$

$$Y(z) = H(z)X(z) = \frac{z^{-1}(z^{-1}+z^{-2})}{(1-0.7z^{-1}-0.12z^{-2})(1-z^{-1})^2} = \frac{z^{-2}(z^{-1}+1)}{(1-0.3z^{-1})(1-0.4z^{-1})(1-z^{-1})^2}$$

$$= \frac{4.76z^{-1}}{(1-z^{-1})^2} + \frac{-12.36}{1-z^{-1}} + \frac{-26.5}{1-0.3z^{-1}} + \frac{38.9}{1-0.4z^{-1}}$$

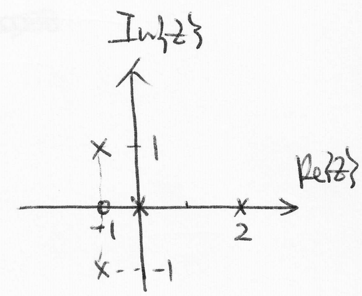
$\Rightarrow y(n) = [4.76n - 12.36 - 26.5(0.3)^n + 38.9(0.4)^n]u[n]$.

$\boxed{3.39}$

$X(z) = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1-pz^{-1})(1-p^*z^{-1})}, \quad p = -\frac{1}{2}+\frac{1}{2}j$

(a) $X_1(z) = z^{-2}X(z^{-1}) = \frac{z^{-2}(1+z)}{(1-\frac{1}{2}z)(1-pz)(1-p^*z)}, \quad ROC = |z|<2$

Zeros: $z=-1, \infty$. poles: $z = 2, \frac{1}{p}, \frac{1}{p^*}, 0$

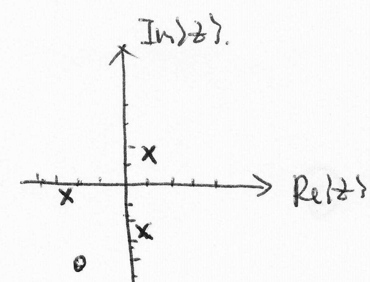


(b) $X_2(z) = X(e^{-j\pi/3}z)$

$$= \frac{1+e^{j\frac{\pi}{3}}z^{-1}}{(1-\frac{1}{2}e^{j\frac{\pi}{3}}z^{-1})(1-pe^{j\frac{\pi}{3}}z^{-1})(1-p^*e^{j\frac{\pi}{3}}z^{-1})}$$

Zeros: $z = -e^{j\frac{\pi}{3}}$ poles: $z = \frac{1}{2}e^{j\frac{\pi}{3}}, pe^{j\frac{\pi}{3}}, p^*e^{j\frac{\pi}{3}}$

ROC is the same as $X(z)$



3.40

Input: $x(n) = (\frac{1}{2})^n u(n) - \frac{1}{4}(\frac{1}{2})^{n-1} u(n-1)$

Output: $y(n) = (\frac{1}{3})^n u(n)$

Thus $X(Z) = \frac{1}{1-\frac{1}{2}Z^{-1}} - \frac{1}{4}\frac{Z^{-1}}{1-\frac{1}{2}Z^{-1}}$, $Y(Z) = \frac{1}{1-\frac{1}{3}Z^{-1}}$

(a) $H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1-\frac{1}{2}Z^{-1}}{(1-\frac{1}{3}Z^{-1})(1-\frac{1}{4}Z^{-1})} = \frac{3}{1-\frac{1}{4}Z^{-1}} - \frac{2}{1-\frac{1}{3}Z^{-1}}$

Thus $h(n) = [3(\frac{1}{4})^n - 2(\frac{1}{3})^n]\, u(n) \Leftarrow$ system is causal.

(b) $H(Z) = \frac{1-\frac{1}{2}Z^{-1}}{(1-\frac{1}{3}Z^{-1})(1-\frac{1}{4}Z^{-1})} = \frac{1-\frac{1}{2}Z^{-1}}{1-\frac{7}{12}Z^{-1}+\frac{1}{12}Z^{-2}}$

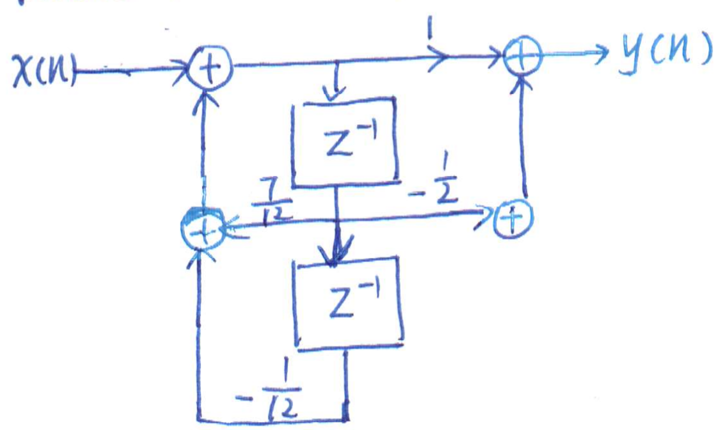
Thus $\frac{Y(Z)}{X(Z)} = \frac{1-\frac{1}{2}Z^{-1}}{1-\frac{7}{12}Z^{-1}+\frac{1}{12}Z^{-2}}$

Thus $Y(Z) - \frac{7}{12}Z^{-1}Y(Z) + \frac{1}{12}Z^{-2}Y(Z) = X(Z) - \frac{1}{2}Z^{-1}X(Z)$

Thus the difference equation is

$y(n) = \frac{7}{12}y(n-1) - \frac{1}{12}y(n-2) + x(n) - \frac{1}{2}x(n-1)$

(c) Direct Form II Realization



(d) The poles of the system $p_1 = \frac{1}{3}$, $p_2 = \frac{1}{4}$ are both inside the unit circle. Thus the system is stable.

$\boxed{3.41}$ Given $H(z) = \dfrac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$

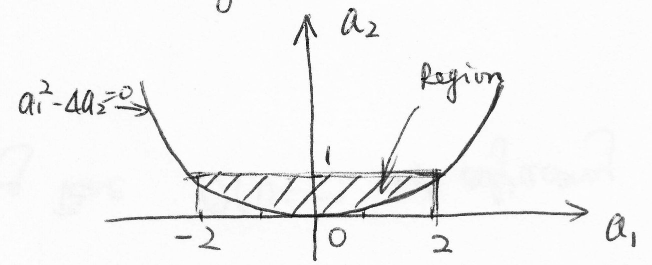
$\Rightarrow$ If $a_1^2 - 4a_2 < 0$, there are two complex poles.

$$p_{1,2} = \frac{-a_1 \pm j\sqrt{4a_2 - a_1^2}}{2}$$

Restrict poles inside the unit circle, we get

$$|p_{1,2}|^2 = \left(\frac{a_1}{2}\right)^2 + \left(\frac{\sqrt{4a_2 - a_1^2}}{2}\right)^2 = \left(\frac{a_1}{2}\right)^2 + \frac{4a_2 - a_1^2}{4} = a_2 < 1$$

Therefore, the region is



$\Rightarrow$ If $a_1^2 - 4a_2 \geq 0$, there are two real poles.

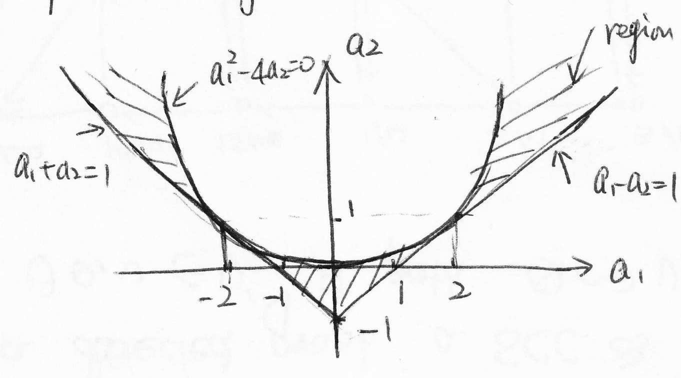
$$p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Restrict $p_{1,2}$ inside the unit circle, we get

$$p_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} < 1 \;\; \&\& \;\; p_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} > -1$$

$\Rightarrow$ $a_1 - a_2 < 1$ && $a_1 + a_2 > 1$

Therefore, the region is



3.51.

$P_1 = -3$, $P_2 = -0.5$, $P_3 = 2$, $Z = 1$.

(a) Thus $H(z) = \dfrac{k(z-1)}{(z+\frac{1}{2})(z+3)(z-2)}$, where k is any nonzero constant.

Since the system is stable, ROC has to include unit circle.
Therefore ROC is $\frac{1}{2} < |z| < 2$.

(b) If the system is causal, the ROC is the smallest circle encompassing all the poles, thus ROC is $|z| > 3$.
Since the system is causal and the poles $P_1 = -3$, $P_3 = 2$ lie outside of the unit circle, we have the system is not stable.

(c) (1) causal: ROC: $|z| > 3$

(2) anti-causal: ROC: $|z| < 3$

(3) noncausal: ROC: $\frac{1}{2} < |z| < 2$.

(4) noncausal: ROC: $2 < |z| < 3$.

3.53

The answer is no. For the given system, $h_1(n) = a^n u(n)$. Its Z-transform is $H_1(z) = \frac{1}{1-az^{-1}}$, $|a|<1$. This system is causal and stable. However, when $h_2(n) = a^n u(n+3)$, its Z-transform is $H_2(z) = \frac{a^{-3}z^3}{1-az^{-1}}$, $|a|<1$. The system is stable but is not causal.