Discrete-Time Fourier Magnitude and Phase

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Complex Nature of $X(\omega)$

Recall, Fourier Transform:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \in \mathbb{C}$$

and Inverse Fourier Transform:

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-0}^{0} X(\omega)e^{j\omega n} d\omega + \frac{1}{2\pi} \int_{0}^{\pi} X(\omega)e^{j\omega n} d\omega$$

Note: If $x(n)$ is real, then the imaginary part of the negative frequency sinusoids (i.e., $e^{j\omega n}$ for $\omega<0$) cancel out the imaginary part of the positive frequency sinusoids (i.e., $e^{j\omega n}$ for $\omega>0$)

Magnitude and Phase of $X(\omega)$

- Rectangular coordinates: rarely used in signal processing
  $$X(\omega) = X_R(\omega) + j X_I(\omega)$$
  where $X_R(\omega), X_I(\omega) \in \mathbb{R}$.

- Polar coordinates: more intuitive way to represent frequency content
  $$X(\omega) = |X(\omega)| e^{j\Theta(\omega)}$$
  where $|X(\omega)|, \Theta(\omega) = \angle X(\omega) \in \mathbb{R}$. 

- $|X(\omega)|$: determines the relative presence of a sinusoid $e^{j\omega n}$ in $x(n)$

- $\Theta(\omega) = \angle X(\omega)$: determines how the sinusoids line up relative to one another to form $x(n)$
Magnitude and Phase of $X(\omega)$

\[
x(n) = \frac{1}{2\pi} \int_{2\pi} |X(\omega)| e^{j\Theta(\omega)} e^{j\omega n} d\omega
\]

- Recall, $e^{j(\omega n + \Theta(\omega))} = \cos(\omega n + \Theta(\omega)) + j\sin(\omega n + \Theta(\omega))$.
- The larger $|X(\omega)|$ is, the more prominent $e^{j\omega n}$ is in forming $x(n)$.
- $\Theta(\omega) = \arg X(\omega)$ determines the relative phases of the sinusoids (i.e. how they line up with respect to one another).

Example: audio information signal

- An audio signal is represented by a real function $x(n)$.
- The function $x(-n)$ represents playing the audio signal backwards.
- Since $x(n)$ is real:

\[
X(\omega) = X^*(-\omega)
\]
\[
|X(\omega)| = |X^*(-\omega)| = |X(-\omega)| \quad \text{since} \quad |c| = |c^*| \quad \text{for} \quad c \in \mathbb{C}
\]

- Therefore,

\[
|X(\omega)| = |X(-\omega)|
\]

That is, the FT magnitude is even for $x(n)$ real.

Magnitude versus Phase

Q: Which is more important for a given signal?

- Does one component (magnitude or phase) contain more information than another?
- When filtering, if we had to preserve one component (magnitude or phase) more, which one would it be?

Example: audio information signal

- Recall, $x(n) \leftrightarrow X(\omega)$ $x(-n) \leftrightarrow X(-\omega)$
- Therefore,

\[
|X(\omega)| = |X(-\omega)|
\]

Therefore, the magnitude of the FT of an audio signal played forward and backward is the same!
Example: grayscale still images

- A still image can be considered a two-dimensional signal: \( x(n_1, n_2) \) where \( n_1 \) represents the horizontal dimension and \( n_2 \) represents the vertical dimension.

Discrete-Time Fourier Magnitude and Phase

Intensity Images

- **discrete-space** and **continuous-amplitude** image consisting of intensity (grayscale) values

- \( x(n_1, n_2) \) is a two-dimensional signal representing the grayscale value at location \((n_1, n_2)\) where:
  - \( 0 \leq n_1 \leq N_1 \) and \( 0 \leq n_2 \leq N_2 \)
  - \( x(n_1, n_2) = 0 \) represents black
  - \( x(n_1, n_2) = 1 \) represents white
  - \( 0 < x(n_1, n_2) < 1 \) represents proportional gray-value

Analog Intensity Images

- \( x(n_1, n_2) \) can be displayed as an intensity image or as a mesh graph

Example: grayscale still images

- The Fourier transform \( x(n_1, n_2) \) has two frequency variables: \( \omega_1 \) and \( \omega_2 \) and is given by:

\[
X(\omega_1, \omega_2) = \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} x(n_1, n_2) e^{-j(\omega_1 n_1 + \omega_2 n_2)} \in \mathbb{C}
\]

- Typically, we consider the magnitude and phase of \( X(\omega_1, \omega_2) \):

\[
|X(\omega_1, \omega_2)| \quad \text{and} \quad \angle X(\omega_1, \omega_2)
\]
Example: $x(n_1, n_2)$

Example: $|X(\omega_1, \omega_2)|$

Example: $\Theta(\omega_1, \omega_2) = \angle X(\omega_1, \omega_2)$
Reconstruction using magnitude only
Top Left Photo: Ralph’s magnitude is the same, Phase = 0
Top Right Photo: Meg’s magnitude is the same, Phase = 0

Reconstruction using phase only
Top Left Photo: Ralph’s magnitude normalized to one, Phase is the same
Top Right Photo: Meg’s magnitude normalized to one, Phase is the same

Reconstruction swapping magnitude and phase of the images.
Top Left Photo: Ralph’s phase + Meg’s magnitude
Top Right Photo: Meg’s phase + Ralph’s magnitude

Magnitude versus Phase

Q: Which is more important for a given signal? A: Phase.

- Does one component (magnitude or phase) contain more information than another? A: Yes, typically phase.
- When filtering, if we had to preserve on component (magnitude or phase) more, which one would it be? A: It is important to preserve phase during filtering.