http://www.comm.utoronto.ca/~dkundur/course/discrete-time-systems/

HOMEWORK #1 - SOLUTIONS

1.2

For a sinusoid of the form $x(n) = A\cos(\omega \cdot n + \theta)$, the frequency can be expressed as $f = \dfrac{\omega}{2\pi}$.

Definition *B1* from the textbook states:

"A discrete-time sinusoid is periodic only if its frequency f_0 is a rational number."

Furthermore, the fundamental period of a sinusoid when it exists is equal to *N* where $f_0 = \dfrac{k}{N}$ and *k, N* are relatively prime.

(a) $\cos(0.01\pi \cdot n)$

$f = \dfrac{\omega}{2\pi} = \dfrac{0.01\pi}{2\pi} = \dfrac{1}{200}$ is rational and (1,200) are relatively prime.

→ $\cos(0.01\pi \cdot n)$ is periodic with fundamental period N = 200.

(b) $\cos\left(\pi \cdot \dfrac{30}{105} n\right)$

$f = \dfrac{\omega}{2\pi} = \dfrac{30\pi}{105} \cdot \dfrac{1}{2\pi} = \dfrac{1}{7}$ is rational and (1,7) are relatively prime.

→ $\cos\left(\pi \cdot \dfrac{30}{105} n\right)$ is periodic with fundamental period $N = 7$.

(c) $\cos(3\pi \cdot n)$

$f = \dfrac{\omega}{2\pi} = \dfrac{3\pi}{2\pi} = \dfrac{3}{2}$ is rational and (3,2) are relatively prime.

→ $\cos(3\pi \cdot n)$ is periodic with fundamental period N = 2.

(d) $\sin(3 \cdot n)$

$f = \dfrac{\omega}{2\pi} = \dfrac{3}{2\pi} = \dfrac{3}{2\pi}$ is not rational.

→ $\sin(3 \cdot n)$ is non-periodic.

(d) $\sin\left(\pi \cdot \dfrac{62}{10} n\right)$

$f = \frac{\omega}{2\pi} = \frac{62\pi}{10} \cdot \frac{1}{2\pi} = \frac{31}{10}$ is rational and (31,10) are relatively prime.

$\rightarrow \sin\left(\pi \cdot \frac{62}{10} n\right)$ is periodic with fundamental period N = 10.

1.4

(a) As mentioned previously the fundamental period of a sinusoid when it exists is equal to N where $f_0 = \frac{\omega}{2\pi} = \frac{k}{N}$ and *k, N* are relatively prime.

For $s_k(n) = \exp\left(j\frac{2\pi \cdot kn}{N}\right), \quad k = 0,1,2,...$, we have $f = \frac{\omega}{2\pi} = \frac{k}{N}$.

Suppose now that *k* and *N* are not relatively prime, i.e. $\exists \alpha$ such that $\alpha = GCD(k, N)$ and $k = k' \cdot \alpha$ and $N = N' \cdot \alpha$ where *k'* and *N'* are relatively prime. We can now rewrite:
$f_0 = \frac{\omega}{2\pi} = \frac{k}{N} = \frac{k' \cdot \alpha}{N' \cdot \alpha} = \frac{k'}{N'}$.

As *k'* and *N'* are relatively prime, the fundamental period of the signals
$s_k(n) = \exp\left(j\frac{2\pi \cdot kn}{N}\right), \quad k = 0,1,2,...$ is $N_p = N' = \frac{N}{GCD(k,N)}$

(b) Fundamental period for $N = 7$

N	7	7	7	7	7	7	7	7
k	0	1	2	3	4	5	6	7
GCD(k,N)	7	1	1	1	1	1	1	7
N_p	1	7	7	7	7	7	7	1

(c) Fundamental period for $N = 16$

N	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16	16
k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
GCD(k,N)	16	1	2	1	4	1	2	1	8	1	2	1	4	1	2	1	16
N_p	1	16	8	16	4	16	8	16	2	16	8	16	4	16	8	16	1

1.6

Because $x(n)$ is a discrete-time sinusoid obtained by sampling at a rate $F_s = \frac{1}{T}$ a continuous-time sinusoid $x_a(t)$ with fundamental period $T_p = \frac{1}{F_0}$, we find $x(n)$ as follows:

$$x_a(t) = A\cos(2\pi F_0 t + \theta) \Rightarrow x(n) = x_a(nT) = A\cos(2\pi \cdot F_0 nT + \theta)$$
$$= A\cos\left(2\pi \cdot \frac{F_0}{F_s} n + \theta\right)$$
$$= A\cos\left(2\pi \cdot \frac{T}{T_p} n + \theta\right)$$

(a) If $\frac{T}{T_p} = f$ is rational *i.e.,* $\frac{T}{T_p} = \frac{k}{N}$, we have straight from definition *B1* from the textbook that $x(n)$ is periodic.

(b) $x(n)$ is periodic implies f rational i.e., $f = \frac{k}{N}$. *N* here represents the period in samples which means that to get the period in seconds *N* must be multiplied by *T*. Then we get $T_d = N \cdot T = \frac{k}{f} T$ but $f = \frac{T}{T_p}$ and therefore the final result:

$$T_d = k \frac{T_p}{T} T = kT_p$$

(c) Supposing the fundamental period T_p of $x(n)$ verifies $T_d = kT_p$, it implies that $T_d = NT = kT_p$ or $\frac{T}{T_p} = \frac{k}{N} = f$. f is therefore rational i.e., from the result obtained in (a), $x(n)$ is periodic.

1.7

(a) To avoid any aliasing problem, it is known that a signal must be sampled as at least twice its maximum frequency : $F_s \geq 2 \cdot F_{\max}$. In this case F_{max} = 10kHz and therefore the sample frequency should be at least 20kHz.

$$F_s \geq 20kHz$$

(b) If $F_s = 8$kHz and $F_1 = 5$kHz the sampled signal becomes

$$x(n) = \cos\left(2\pi \cdot \frac{F_1}{F_s} n\right) = \cos(2\pi \cdot f \cdot n) = \cos\left(\pi \frac{5}{4} n\right) \quad or \quad \cos\left(\pi \frac{3}{4} n\right)$$

From the equality $F_0 = f \cdot F_s$, we obtain $F_0 = 3$ kHz and we can therefore conclude that for $F_s = 8$ kHz, the frequency $F_1 = 5$ kHz is aliased to 3 kHz.

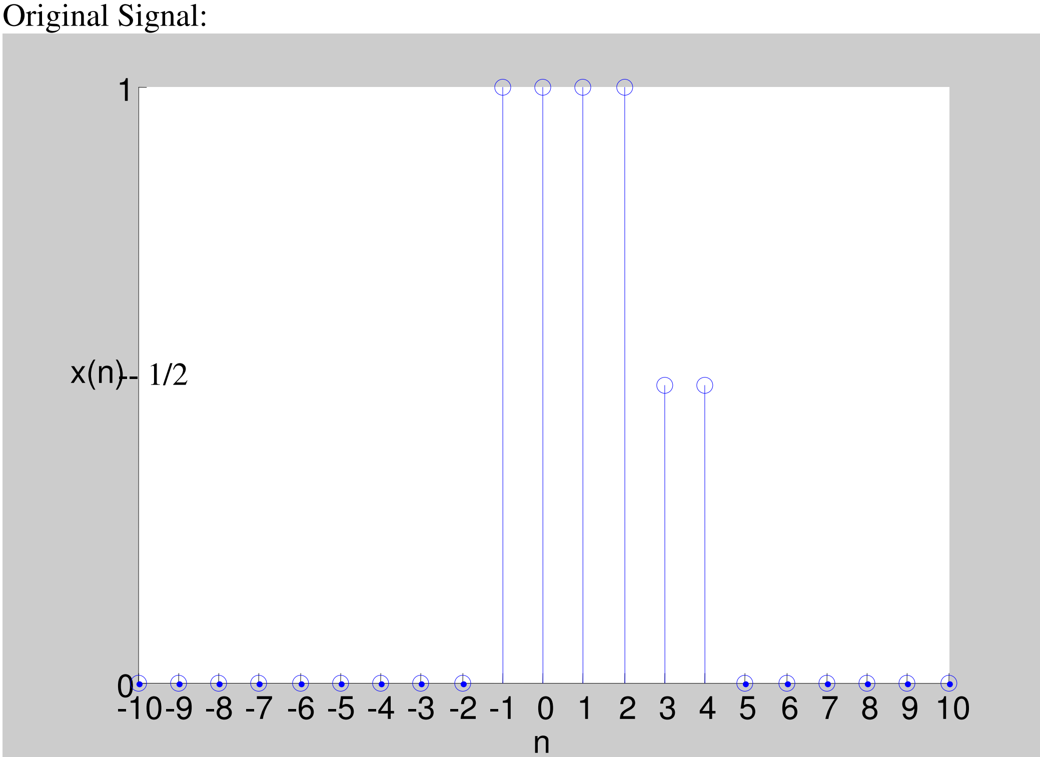
(c) If $F_s = 8$kHz and $F_2 = 9$kHz the sampled signal becomes

$$x(n) = \cos\left(2\pi \cdot \frac{F_2}{F_s} n\right) = \cos(2\pi \cdot f \cdot n) = \cos\left(\pi \frac{9}{4} n\right) \quad or \quad \cos\left(\pi \frac{1}{4} n\right)$$

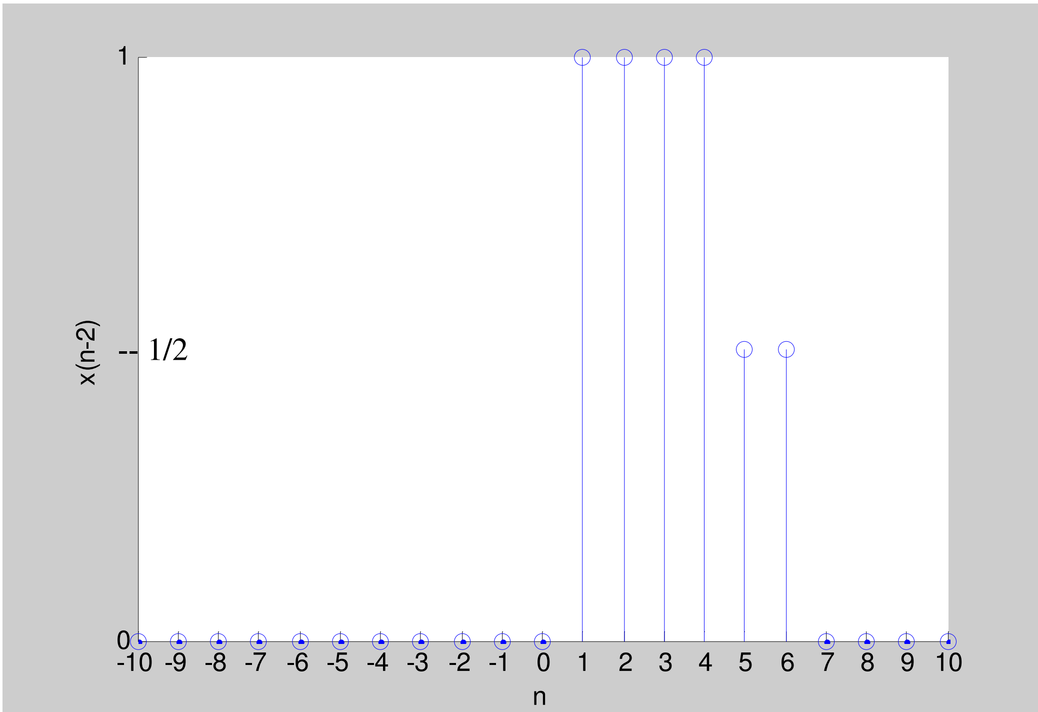
From the equality $F_0 = f \cdot F_s$, we obtain $F_0 = 1$ kHz and we can therefore conclude that for $F_s = 8$ kHz, the frequency $F_2 = 9$ kHz is aliased to 1 kHz.

2.2

Original Signal:

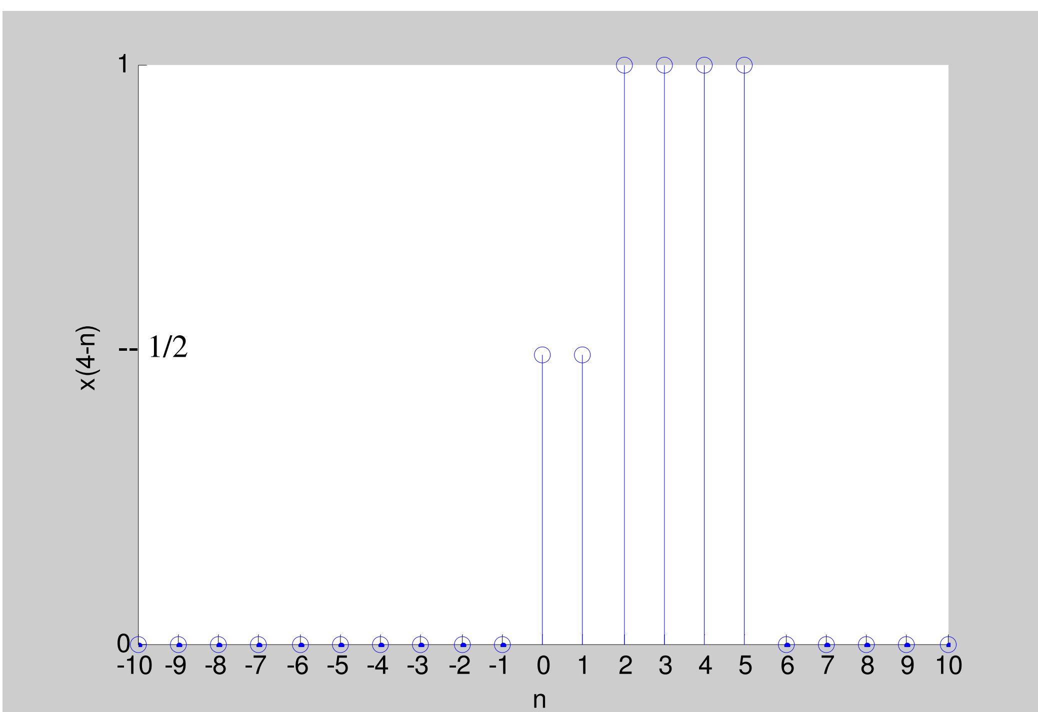


(a)



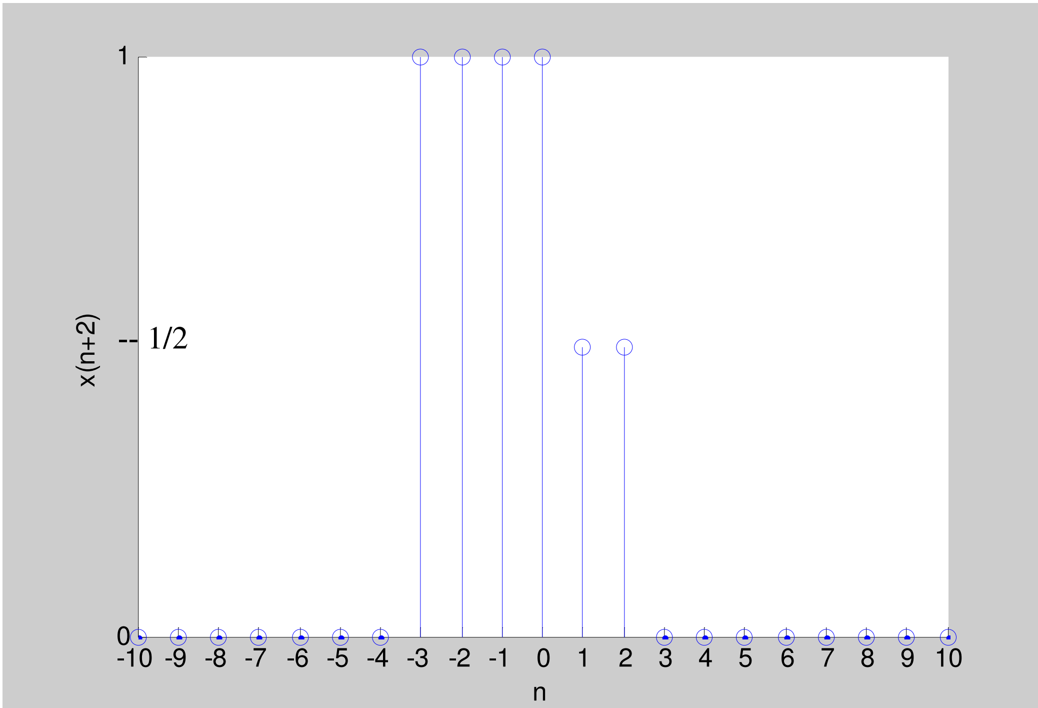
1
1/2
0
x(n-2)
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
n

(b)

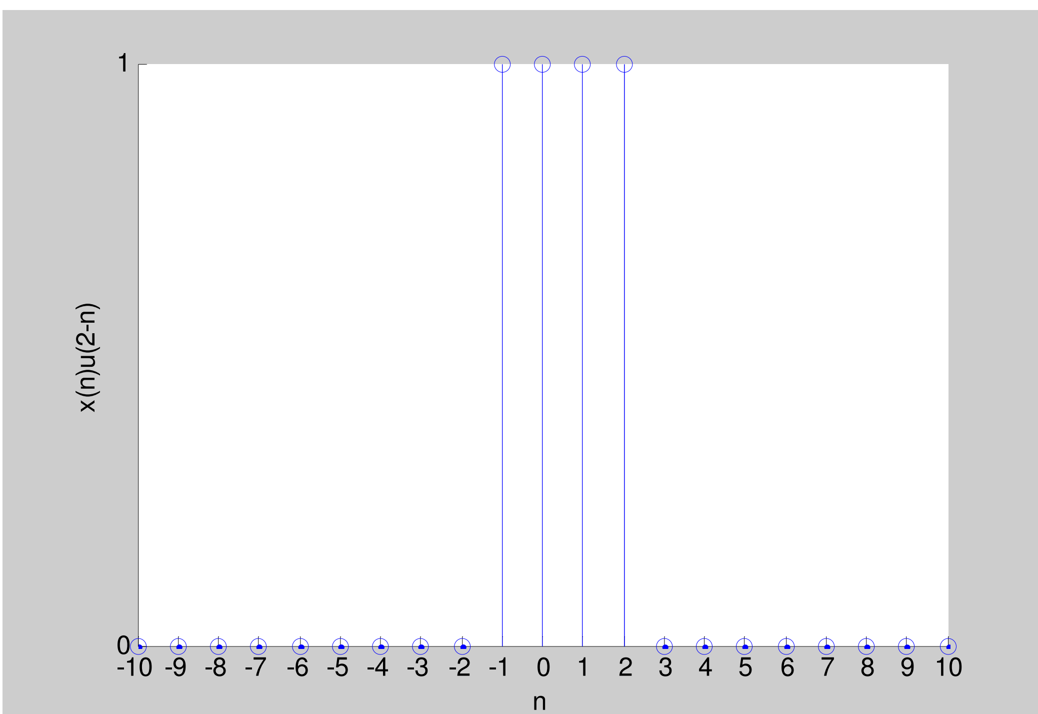


1
1/2
0
x(4-n)
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
n

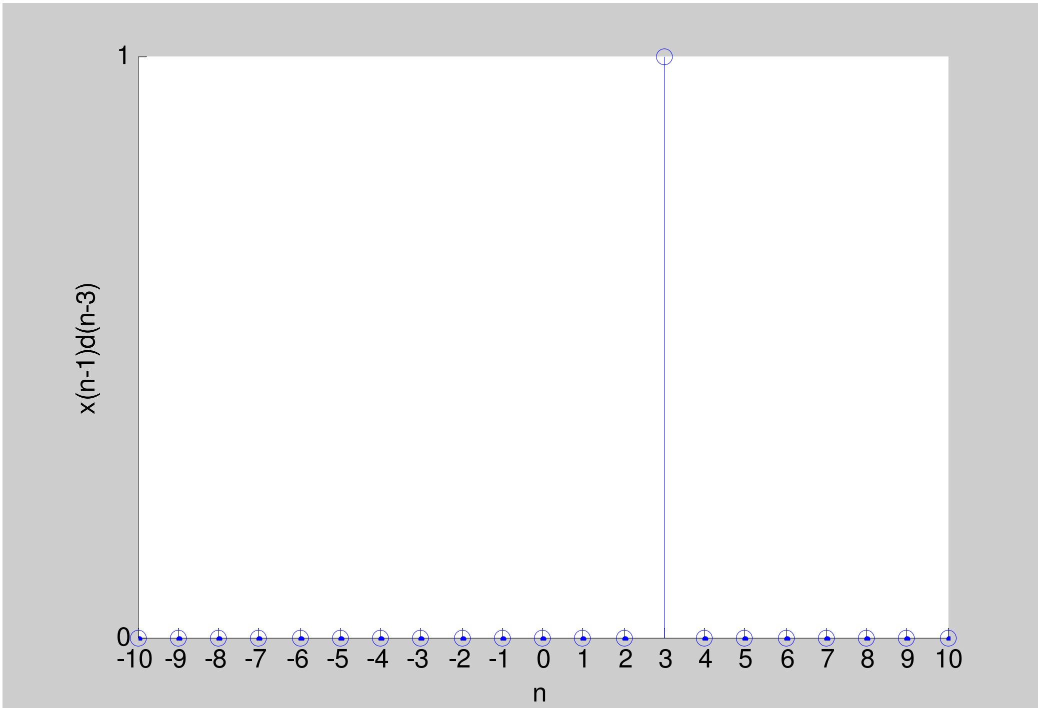
(c)


1
1/2
0
x(n+2)
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
n

(d)

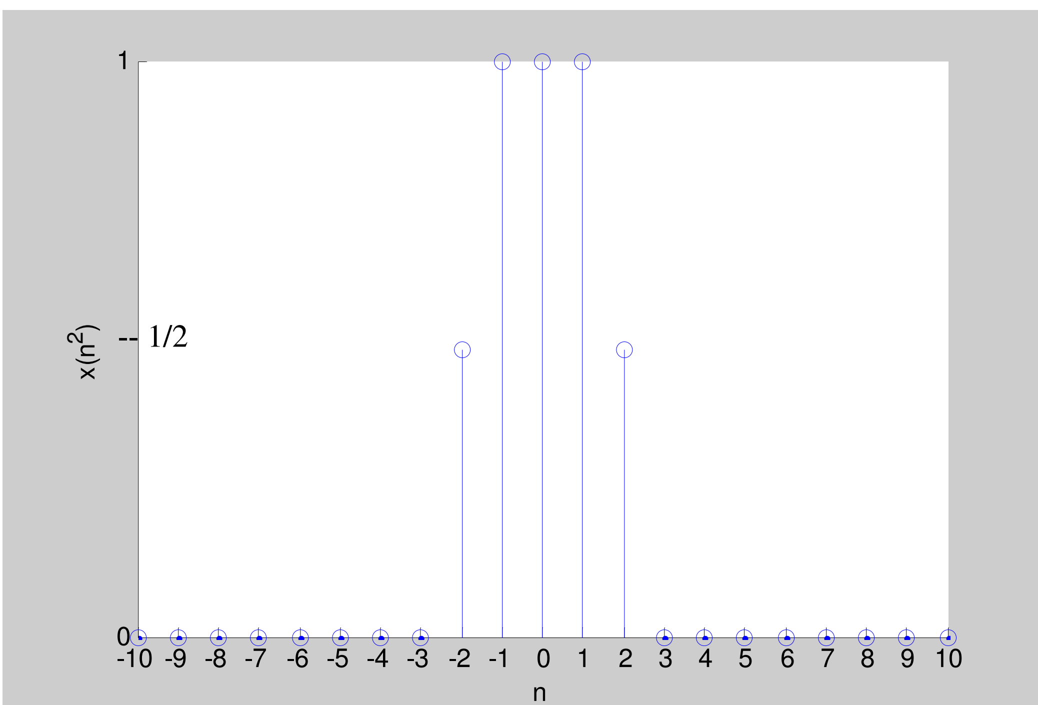

1
0
x(n)u(2-n)
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
n

(e)



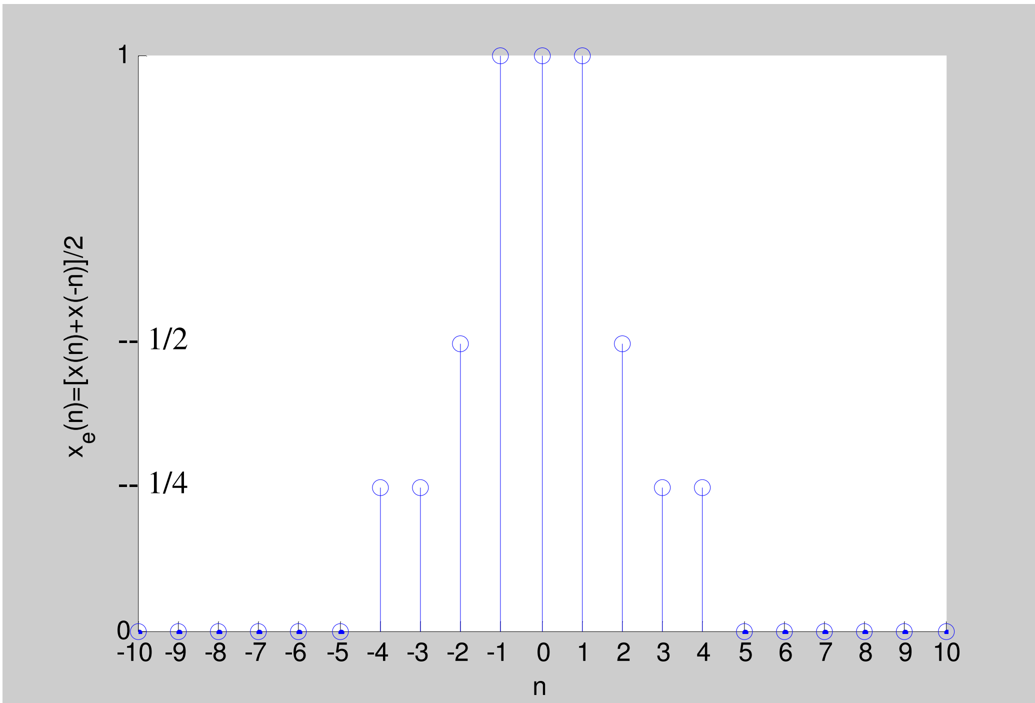
1
0
x(n-1)d(n-3)
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
n

(f)



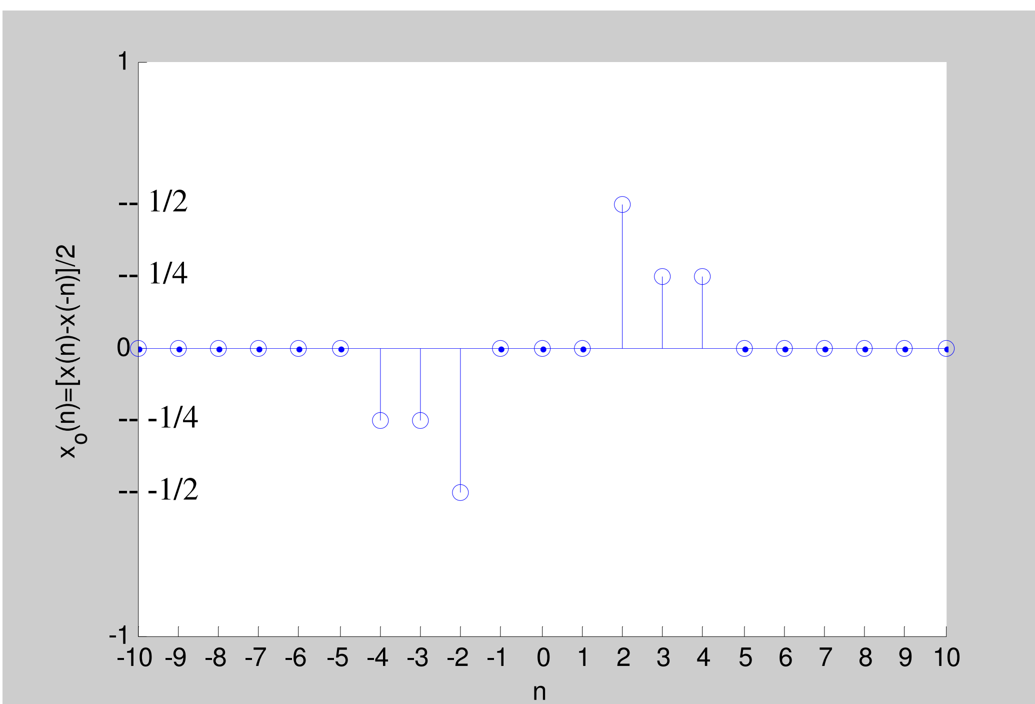
1
-- 1/2
0
x(n²)
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
n

(g)



1
1/2
1/4
0
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
n
$x_e(n)=[x(n)+x(-n)]/2$

(h)



1
1/2
1/4
0
-1/4
-1/2
-1
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
n
$x_o(n)=[x(n)-x(-n)]/2$

2.4

The even and odd parts of a signal $x(n)$ can be found using the following formulas:

$$x_{even}(n) = \frac{[x(n) + x(-n)]}{2} \qquad x_{odd}(n) = \frac{[x(n) - x(-n)]}{2}$$

Where $x_{even}(n)$ being the even part verifies $x_{even}(-n) = x_{even}(n)$ and $x_{odd}(n)$ being the odd part verifies $x_{odd}(-n) = -x_{odd}(n)$.

Therefore any $x(n)$ can be written as $x(n) = x_{even}(n) + x_{odd}(n)$ i.e., any signal can be decomposed into an even and an odd component and this decomposition is unique.

Example:

$$x(n) = \{2 \quad 3 \quad \underset{\uparrow}{4} \quad 5 \quad 6\}$$

$$x(-n) = \{6 \quad 5 \quad \underset{\uparrow}{4} \quad 3 \quad 2\}$$

$$x_{even}(n) = \{4 \quad 4 \quad \underset{\uparrow}{4} \quad 4 \quad 4\}$$

$$x_{odd}(n) = \{-2 \quad -1 \quad \underset{\uparrow}{0} \quad 1 \quad 2\}$$

2.5

The energy of a real-valued signal $x(n)$ is defined as:

$$E = \sum_{n=-\infty}^{\infty} x^2(n)$$

From the previous problem, it can also be expressed by a combination of the even and odd parts of the signal as follows:

$$E = \sum_{n=-\infty}^{\infty} x^2(n) = \sum_{n=-\infty}^{\infty} [x_{even}(n) + x_{odd}(n)]^2$$

$$= \sum_{n=-\infty}^{\infty} x_{even}^2(n) + \sum_{n=-\infty}^{\infty} x_{odd}^2(n) + 2 \cdot \sum_{n=-\infty}^{\infty} x_{even}(n) \cdot x_{odd}(n)$$

As it can be seen, the term $2 \cdot \sum_{n=-\infty}^{\infty} x_{even}(n) \cdot x_{odd}(n)$ in the previous equation should be equal to zero if we believe the problem formulation. Let's try to prove it. Because the summation is symmetric in *n*, we have:

$$2 \cdot \sum_{n=-\infty}^{\infty} x_{even}(n) \cdot x_{odd}(n) = 2 \cdot \sum_{m=-\infty}^{\infty} x_{even}(-m) \cdot x_{odd}(-m)$$

We know that $x_{even}(n)$ being the even part verifies $x_{even}(-n) = x_{even}(n)$ and $x_{odd}(n)$ being the odd part verifies $x_{odd}(-n) = -x_{odd}(n)$, therefore we can write:

$$2 \cdot \sum_{m=-\infty}^{\infty} x_{even}(-m) \cdot x_{odd}(-m) = 2 \cdot \sum_{m=-\infty}^{\infty} x_{even}(m) \cdot (-x_{odd}(m))$$

$$= -2 \cdot \sum_{n=-\infty}^{\infty} x_{even}(n) \cdot x_{odd}(n)$$

We then obtain the equality $2 \cdot \sum_{n=-\infty}^{\infty} x_{even}(n) \cdot x_{odd}(n) = -2 \cdot \sum_{n=-\infty}^{\infty} x_{even}(n) \cdot x_{odd}(n)$ which proves that $2 \cdot \sum_{n=-\infty}^{\infty} x_{even}(n) \cdot x_{odd}(n) = 0$ and finally:

$$E = \sum_{n=-\infty}^{\infty} x^2(n) = \sum_{n=-\infty}^{\infty} x_{even}^2(n) + \sum_{n=-\infty}^{\infty} x_{odd}^2(n)$$

2.7

Recall that:

- A system is static if the output at time *n* depends only on inputs at the same time *n*.

- A system $y(n) = F[x(n)]$ is linear if and only if it verifies:

$$F[a_1 x_1(n) + a_2 x_2(n)] = a_1 F[x_1(n)] + a_2 F[x_2(n)]$$

- A system is time invariant if and only if $x(n) \xrightarrow{F} y(n) \Rightarrow x(n-k) \xrightarrow{F} y(n-k), \quad \forall k$ time shift.

- A system is causal if it depends only on past or present inputs.
- A system is (BIBO) stable if when the input $x(n)$ is bounded, the output $y(n)$ is also bounded.

(a) $y(n) = \cos[x(n)]$

- the output at time n depends only on the input at time n, the system is static.
- the system is nonlinear, e.g., $x(n) = \pi \Rightarrow y(n) = \cos[\pi] = -1$ but
$x_1(n) = \frac{1}{2}x(n) = \frac{\pi}{2} \Rightarrow y_1(n) = \cos\left[\frac{\pi}{2}\right] = 0 \neq \frac{1}{2}y(n) = -\frac{1}{2}$
- the system is time invariant :
$n \to n-k \Rightarrow y(n) = \cos[x(n)] \to y(n-k) = \cos[x(n-k)], \quad \forall k$
- the system is static and therefore is also causal.
- the output of the system is always bounded therefore it is obviously stable.

→ The system $y(n) = \cos[x(n)]$ is static, nonlinear, time invariant, causal and stable.

(g) $y(n) = |x(n)|$

- the output at time n depends only on the input at time n, the system is static.
- the system is nonlinear, e.g.,
$x_1(n) = 1, \quad x_2(n) = -1 \Rightarrow y(n) = |x_1(n) + x_2(n)| = 0 \neq y_1(n) + y_2(n) = |x_1(n)| + |x_2(n)| = 2$
- the system is time invariant : $n \to n-k \Rightarrow y(n) = |x(n)| \to y(n-k) = |x(n-k)|, \quad \forall k$
- the system is static and therefore is also causal.
- $|x(n)| \leq M < \infty \Rightarrow y(n) = |x(n)| \leq |y(n)| \leq M < \infty$ therefore it is obviously stable.

→ The system $y(n) = |x(n)|$ is static, nonlinear, time invariant, causal and stable.

(j) $y(n) = x(2n)$

- the output at time n depends on inputs at time future time 2n, the system is dynamic.
- the system is linear (trivial)
  $y_1(n) = x_1(2n), \quad y_2(n) = x_2(2n) \Rightarrow y(n) = [a_1 x_1(2n) + a_2 x_2(2n)] = a_1 y_1(n) + a_2 y_2(n)$
- the system is time variant : $y(n-k) = x(2(n-k)) = x(2n-2k) \neq x(2n-k)$
- the system depends on future inputs and therefore is noncausal.
- $|x(n)| \le M < \infty \Rightarrow y(n) = x(2n) \le |y(n)| \le M < \infty$ therefore it is obviously stable.

→ The system $y(n) = x(2n)$ is dynamic, linear, time variant, noncausal and stable.

(k) $y(n) = \begin{cases} x(n) & if\ x(n) \ge 0 \\ 0 & if\ x(n) < 0 \end{cases}$

- the output at time n depends only on the input at time n, the system is static.
- the system is nonlinear $x_1(n) = 2, \quad x_2(n) = -1 \Rightarrow y_1(n) = 2, \quad y_2(n) = 0$ but
  $y(n) = F[x_1(n) + x_2(n)] = F[1] = 1 \neq y_1(n) + y_2(n) = 2$
- the system is time invariant :
  $n \to n-k \Rightarrow y(n) = \begin{cases} x(n) & if\ x(n) \ge 0 \\ 0 & if\ x(n) < 0 \end{cases} \to y(n-k) = \begin{cases} x(n-k) & if\ x(n-k) \ge 0 \\ 0 & if\ x(n-k) < 0 \end{cases}, \quad \forall k$
- the system is static and therefore is also causal.
- $|x(n)| \le M < \infty \Rightarrow |y(n)| \le |x(n)| \le M < \infty$ therefore it is obviously stable.

→ The system $y(n) = \begin{cases} x(n) & if\ x(n) \ge 0 \\ 0 & if\ x(n) < 0 \end{cases}$ is static, nonlinear, time invariant, causal and stable.