http://www.comm.utoronto.ca/~dkundur/course/discrete-time-systems/

HOMEWORK #2 - SOLUTIONS

2.13

The input-output equation of a relaxed LTI system is known to be

$$y(m) = \sum_{n=-\infty}^{\infty} x(n) \cdot h(m-n) = \sum_{n=-\infty}^{\infty} x(m-n) \cdot h(n)$$

To prove that this system is BIBO stable we must prove that a bounded input implies a bounded output. Supposing that $|x(k)| \leq M_x < \infty, \quad \forall k$, we can write:

$$|y(m)| = \left| \sum_{n=-\infty}^{\infty} x(m-n) \cdot h(n) \right| \leq \sum_{n=-\infty}^{\infty} |x(m-n)| \cdot |h(n)| \leq \sum_{n=-\infty}^{\infty} M_x \cdot |h(n)| = M_x \cdot \sum_{n=-\infty}^{\infty} |h(n)|$$

It is now obvious that the output $y(m)$ is bounded if and only if the term $\sum_{n=-\infty}^{\infty} |h(n)|$ is also bounded i.e., $\sum_{n=-\infty}^{\infty} |h(n)| \leq M_h < \infty$.

2.23

The z-transform of the step response $s(n)$ is

$$S(z) = Z\{s(n)\} = Z\{h(n) * u(n)\} = H(z) \cdot \frac{1}{1-z^{-1}} \Rightarrow H(z) = S(z) \cdot 1 - z^{-1}$$

The z-transform of the output $y(n)$ of a LTI system with impulse response $h(n)$ becomes:

$$Y(z) = Z\{y(n)\} = Z\{h(n) * x(n)\} = H(z) \cdot X(z) = S(z) \cdot 1 - z^{-1} \cdot X(z)$$

Taking now the inverse z-transform leads to the final result:

$$y(n) = h(n) * x(n) = [s(n) - s(n-1)] * x(n) = s(n) * x(n) - s(n-1) * x(n)$$

NB. Another way to solve this would be to express $h(n)$ in terms of the unit step function:

$$h(n) = h(n) * \delta(n) = h(n) * [u(n) - u(n-1)] = s(n) - s(n-1)$$

And then we would directly get:

$$y(n) = h(n) * x(n) = [s(n) - s(n-1)] * x(n) = s(n) * x(n) - s(n-1) * x(n)$$

3.6

The hint given is very useful here as $y(n)-y(n-1)=\sum_{k=-\infty}^{n}x(k)-\sum_{k=-\infty}^{n-1}x(k)=x(n)$. The z-transform of this equality leads to $Y(z)-Y(z)\cdot z^{-1}=Y(z)\cdot\left(1-z^{-1}\right)=X(z)$ i.e.,

$$Y(z)=\frac{X(z)}{1-z^{-1}}$$

3.18

The z-transform of $x(n)$ is defined as $Z\{x(n)\}=X(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}$

(d) The z-transform of the signal $x_k(n)$ defined as $x_k(n)=\begin{cases} x\left(\frac{n}{k}\right), & if\ \frac{n}{k}\in \mathrm{Z} \\ 0, & otherwise \end{cases}$

computes:

$$Z\{x_k(n)\}=\sum_{n=-\infty,\frac{n}{k}\in \mathrm{Z}}^{\infty}x\left(\frac{n}{k}\right)z^{-n}$$

By injecting $m=\frac{n}{k}$, the z-transform becomes:

$$Z\{x_k(n)\}=\sum_{n=-\infty,\frac{n}{k}\in \mathrm{Z}}^{\infty}x\left(\frac{n}{k}\right)z^{-n}=\sum_{m=-\infty}^{\infty}x(m)z^{-mk}=\sum_{m=-\infty}^{\infty}x(m)\left(z^{k}\right)^{-m}=X\left(z^{k}\right)$$

3.23

The Taylor series of the exponential function is known to be:

$$e^{x}=\sum_{i=0}^{\infty}\frac{x^{i}}{i!}$$

Therefore we can write:

$$X(z) = e^{z} + e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = 1 + \sum_{n=-\infty}^{\infty} \frac{1}{|n|!} z^{-n}$$

When taking the inverse z-transform the signal $x(n)$ is obtained:

$$x(n) = \delta(n) + \frac{1}{n!}$$

3.40

The system is given by $x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$.

Let choose $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$, then the z-transform of $x_1(n)$ is $Z\{x_1(n)\} = \frac{1}{1 - \frac{1}{2} z^{-1}}$ and

$Z\{x_1(n-1)\} = \frac{z^{-1}}{1 - \frac{1}{2} z^{-1}}$, therefore:

$$X(z) = Z\{x_1(n)\} + \frac{1}{4} Z\{x_1(n-1)\} = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{4} \frac{z^{-1}}{1 - \frac{1}{2} z^{-1}} = \frac{1 - \frac{1}{4} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

We also have:

$$Y(z) == Z\left(\left(\frac{1}{3}\right)^n u(n)\right) = \frac{1}{1 - \frac{1}{3} z^{-1}}$$

(a) The system function $H(z)$ of the desired system can be directly expressed as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{2} z^{-1}}{1 - \frac{1}{3} z^{-1} \cdot 1 - \frac{1}{4} z^{-1}} = \frac{A}{1 - \frac{1}{3} z^{-1}} + \frac{B}{1 - \frac{1}{4} z^{-1}}$$

We solve this equality with the following two equations:

$$H(z)\cdot\left(1-\frac{1}{3}z^{-1}\right)\Bigg|_{z=\frac{1}{3}} = \left.\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-1}}\right|_{z=\frac{1}{3}} = A = -2$$

$$H(z)\cdot\left(1-\frac{1}{4}z^{-1}\right)\Bigg|_{z=\frac{1}{4}} = \left.\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{3}z^{-1}}\right|_{z=\frac{1}{4}} = B = 3$$

And finally we obtain the system function desired:

$$H(z) = \frac{3}{1-\frac{1}{4}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}}$$

The impulse response $h(n)$ is directly found by computing the z-transform of the previous system function $H(z)$:

$$h(n) = \left[3\left(\frac{1}{4}\right)^n - 2\left(\frac{1}{3}\right)^n\right]u(n)$$

(b) From (a) we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\cdot\left(1-\frac{1}{4}z^{-1}\right)} \Leftrightarrow \left(1-\frac{1}{2}z^{-1}\right)\cdot X(z) = \left(1-\frac{1}{3}z^{-1}\right)\cdot\left(1-\frac{1}{4}z^{-1}\right)\cdot Y(z)$$

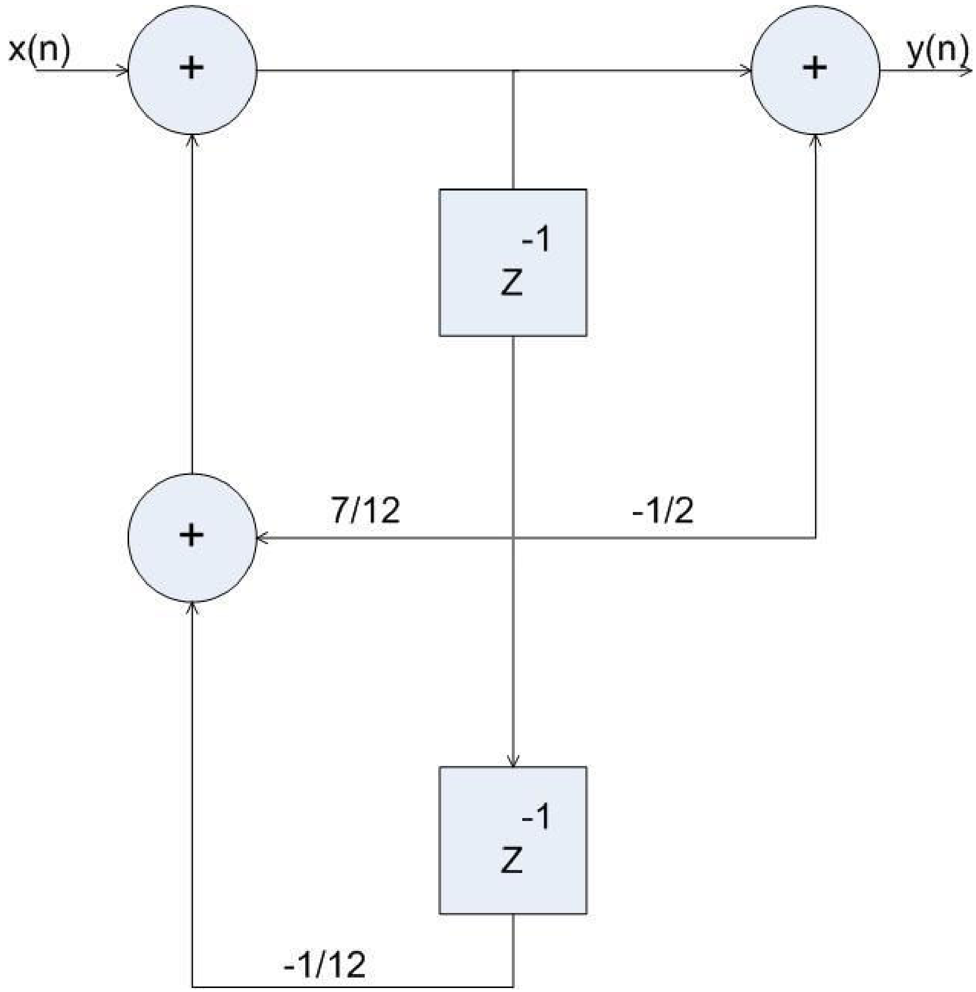
$$= \left(1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}\right)\cdot Y(z)$$

Taking the inverse z-transform of the right-hand side of this equality leads to the desired characterizing difference equation:

$$x(n)-\frac{1}{2}x(n-1) = y(n)-\frac{7}{12}y(n-1)+\frac{1}{12}y(n-2)$$

$$\Leftrightarrow y(n) = \frac{7}{12}y(n-1)-\frac{1}{12}y(n-2)+x(n)-\frac{1}{2}x(n-1)$$

(c) A realization of the desired system could be:



(d) We know that if the poles of the system are inside the unit circle i.e., are less or equal to 1 in absolute value, then the system is stable. The poles for the developed system are $\left\{\frac{1}{3},\frac{1}{4}\right\}$ and are indeed inside the unit circle. Therefore the system is stable.