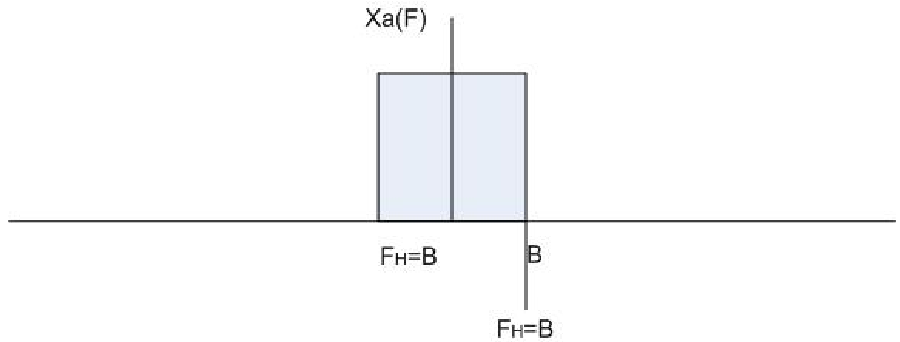
http://www.comm.utoronto.ca/~dkundur/course/discrete-time-systems/

HOMEWORK #4 - SOLUTIONS

6.1

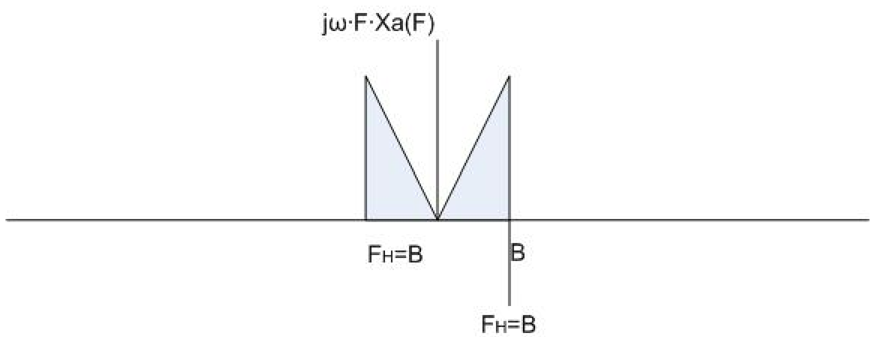
We are given $x_a(t) \leftrightarrow X_a(F) = 0 \quad for \quad |F| > B$. Let's use as an example the following function $X_a(F)$:



We know that the minimum sampling rate F_s in this case would be $F_s = 2F_H = 2B$.

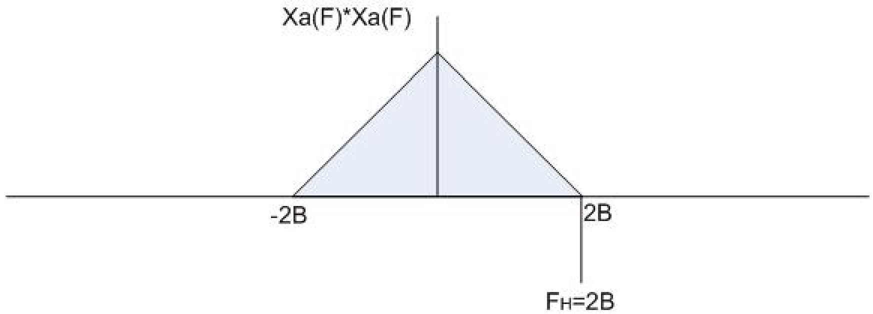
(a) The Fourier transform of the signal $\frac{dx_a(t)}{dt}$ is $j2\pi \cdot F \cdot X_a(F)$. The support for this signal is the same as for $X_a(F)$ and therefore the minimum sampling rate remains the same.

$$F_s = 2B$$



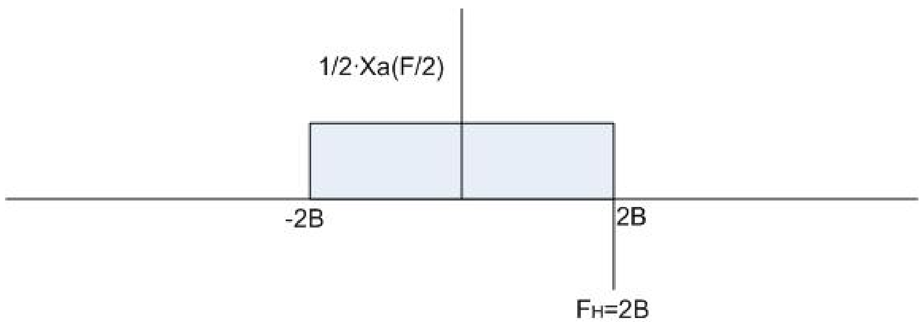
(b) The Fourier transform of the signal $x_a^2(t)$ is $X_a(F) \otimes X_a(F)$. The bandwidth for this signal is twice that of $X_a(F)$ and therefore the minimum sampling rate becomes:

$$F_s = 4B$$



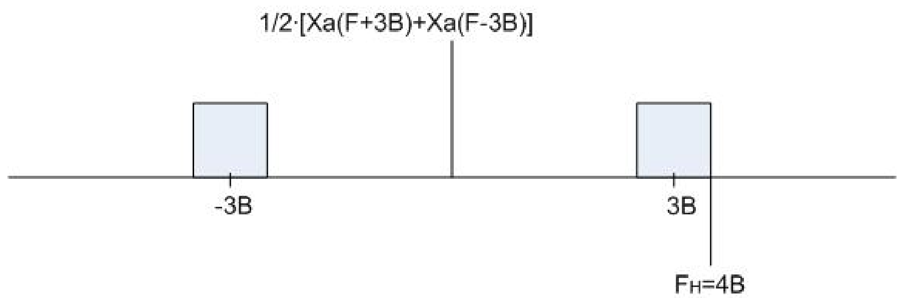
(c) The Fourier transform of the signal $x_a(2t)$ is $\frac{1}{2}\cdot X_a\left(\frac{F}{2}\right)$. The bandwidth for this signal is twice that of $X_a(F)$ and therefore the minimum sampling rate becomes:

$$F_s = 4B$$



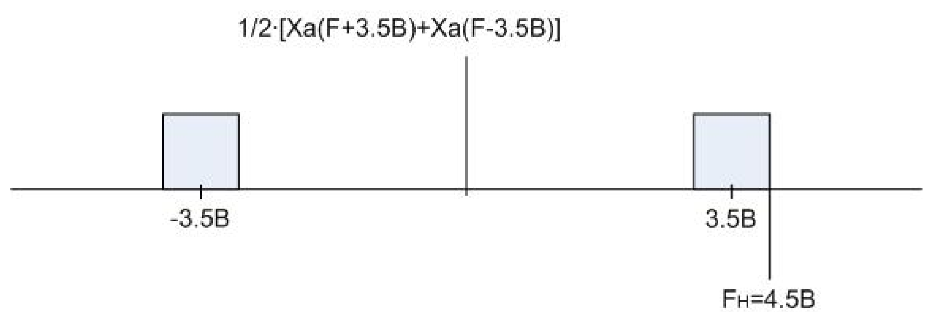
(d) The Fourier transform of the signal $x_a(t)\cos(6\pi Bt)$ is $X_a(F)*\frac{1}{2}[\delta(F-3B)+\delta(F+3B)]$ or $\frac{1}{2}[X_a(F-3B)+X_a(F+3B)]$. The minimum sampling frequency verifies $F_s = \frac{2F_H}{k_{\max}}$ where $k_{\max} = \left\lfloor \frac{F_H}{Bandwitdth} \right\rfloor$. Here, the bandwidth is actually 2B and $F_H = 4B$, therefore $k_{\max} = \left\lfloor \frac{4B}{2B} \right\rfloor = 2$. Finally the minimum sampling frequency is:

$$F_s = \frac{2F_H}{k_{\max}} = \frac{8B}{2} = 4B$$



(e) The Fourier transform of the signal $x_a(t)\cos(7\pi Bt)$ is $X_a(F) * \frac{1}{2}[\delta(F-3.5B)+\delta(F+3.5B)]$ or $\frac{1}{2}[X_a(F-3.5B)+X_a(F+3.5B)]$. The minimum sampling frequency verifies $F_s = \frac{2F_H}{k_{\max}}$ where $k_{\max} = \left\lfloor \frac{F_H}{Bandwitdth} \right\rfloor$. Here, the bandwidth is actually 2B and $F_H = \frac{9}{2}B$, therefore $k_{\max} = \left\lfloor \frac{\frac{9}{2}B}{2B} \right\rfloor = 2$. Finally the minimum sampling frequency is:

$$F_s = \frac{2F_H}{k_{\max}} = \frac{9B}{2} = 4.5B$$



6.10

Recall that to reconstruct the signal without aliasing we need the minimum frequency to verify:

$$F_s = \frac{2F_H}{k_{\max}} \text{ where } k_{\max} = \left\lfloor \frac{F_H}{B} \right\rfloor$$

$\frac{F_H}{B} = \frac{F_c + \frac{B}{2}}{B} = \frac{50 + \frac{20}{2}}{20} = 3$ which is already an integer so $k_{\max} = 3$. Therefore, the minimum sampling frequency is simply $F_s = \frac{2F_H}{k_{\max}} = \frac{2 \cdot 60}{3} = 40Hz$.

6.11

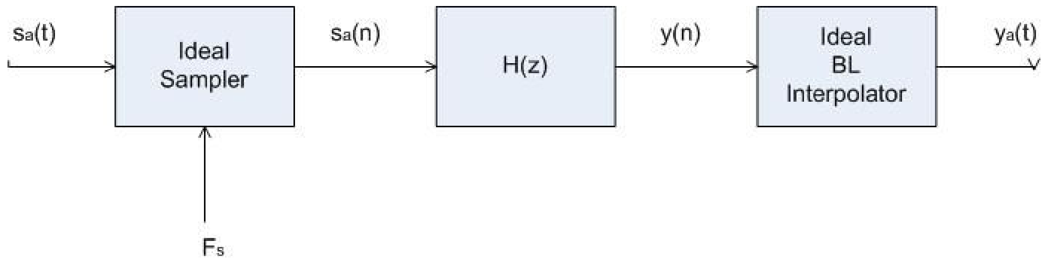
By using the same method as in the previous problem we have:

$\frac{F_H}{B} = \frac{F_c + \frac{B}{2}}{B} = \frac{100 + \frac{12}{2}}{12} = \frac{53}{6}$ which is obviously not an integer and $k_{\max} = \left\lfloor \frac{53}{6} \right\rfloor = 8$.

Therefore, the minimum sampling frequency $F_s = \frac{2F_H}{k_{\max}} = \frac{2 \cdot 106}{8} = 26.5Hz$

6.13

We are given the received analog signal $s_a(t) = x_a(t) + \alpha \cdot x_a(t - \tau)$ with $|\alpha| < 1$ and the following system:



At the end of the sampler the signal $s_a(t)$ will become $s_a(n) = x_a(n) + \alpha \cdot x_a\left(n - \frac{\tau}{T_s}\right)$

where $T_s = \frac{1}{F_s}$.

Taking the Fourier transform we get $S_a(\omega) = X_a(\omega) + \alpha \cdot X_a(\omega) \cdot \exp\left(-j\omega\frac{\tau}{T_s}\right)$ or also,

$\frac{S_a(\omega)}{X_a(\omega)} = 1 + \alpha \cdot \exp\left(-j\omega\frac{\tau}{T_s}\right)$.

If $L = \frac{\tau}{T_s} \in Z$, then we can write the filter $H(z) = \frac{Y_a(z)}{S_a(z)} = \frac{X_a(z)}{S_a(z)} = \frac{1}{1 + \alpha \cdot z^{-L}}$ where $L = \frac{\tau}{T_s}$.