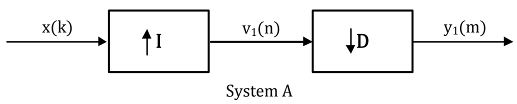
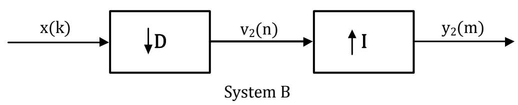
http://www.comm.utoronto.ca/~dkundur/course/discrete-time-systems/

- Homework 7
Solutions

11.9





System A:

$$v_1[n] = \begin{cases} x[\frac{n}{I}] & n = 0, \pm I, \pm 2I, \cdots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\begin{aligned} y_1[m] &= \begin{cases} x[\frac{mD}{I}] & mD = 0, \pm I, \pm 2I, \cdots \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} x[\frac{mD}{I}] & m = 0, \pm \frac{I}{\gcd(D,I)}, \pm \frac{2I}{\gcd(D,I)}, \cdots \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{2}$$

System B:

$$v_2[n] = x[nD] \tag{3}$$

$$\begin{aligned} y_2[m] &= \begin{cases} v_2[\frac{m}{I}] & m = 0, \pm I, \pm 2I, \cdots \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} x[\frac{mD}{I}] & m = 0, \pm I, \pm 2I, \cdots \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4}$$

(a) If $D = I$, we have (from equations 2 and 4):

$$y_1[m] = x[m] \quad m = 0, \pm 1, \pm 2, \cdots \tag{5}$$

$$y_2[m] = \begin{cases} x[m] & m = 0, \pm I, \pm 2I, \cdots \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Thus, $y_1[m]$ and $y_2[m]$ are generally not equal.

(b) If $\gcd(D, I) = 1$, it is plain from equations 2 and 4 that $y_1[m] = y_2[m]$.

Conversely, if $y_1[m] = y_2[m]$, the sets $\{0, \pm\frac{I}{\gcd(D,I)}, \pm\frac{2I}{\gcd(D,I)}, \cdots\}$ and $\{0, \pm I, \pm 2I, \cdots\}$ are equal. This implies $\gcd(D, I) = 1$.

11.11

(a)

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n) z^{-n} \\ &= \left\{\cdots + h(-4)z^4 + h(-2)z^2 + h(0) + h(2)z^{-2} + h(4)z^{-4} + \cdots\right\} \\ &\quad + \left\{\cdots + h(-3)z^3 + h(-1)z^1 + h(1)z^1 + h(3)z^{-3} + \cdots\right\} \\ &= \sum_{n=-\infty}^{\infty} h(2n) z^{-2n} + \sum_{n=-\infty}^{\infty} h(2n+1) z^{-(2n+1)} \\ &= \sum_{n=-\infty}^{\infty} h_0(n) z^{-2n} + z^{-1} \sum_{n=-\infty}^{\infty} h_1(n) z^{-2n} \\ &= H_0(z^2) + H_1(z^2) \end{aligned} \tag{1}$$

where,

$$H_0(z) = \sum_{n=-\infty}^{\infty} h(2n) z^{-n} \tag{2}$$

$$H_1(z) = \sum_{n=-\infty}^{\infty} h(2n+1)z^{-n} \tag{3}$$

(b)

$$\begin{aligned}
H(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} \\
&= \sum_{n=-\infty}^{\infty} h(nD)z^{-nD} + \sum_{n=-\infty}^{\infty} h(nD+1)z^{-(nD+1)} + \cdots + \sum_{n=-\infty}^{\infty} h(nD+D-1)z^{-(nD+D-1)} \\
&= \sum_{n=-\infty}^{\infty} h(nD)z^{-nD} + z^{-1}\sum_{n=-\infty}^{\infty} h(nD+1)z^{-nD} + \cdots + z^{-(D-1)}\sum_{n=-\infty}^{\infty} h(nD+D-1)z^{-nD} \\
&= \sum_{n=-\infty}^{\infty} h_0(n)z^{-nD} + z^{-1}\sum_{n=-\infty}^{\infty} h_1(n)z^{-nD} + \cdots + z^{-(D-1)}\sum_{n=-\infty}^{\infty} h_{D-1}(n)z^{-nD} \\
&= H_0(z^D) + H_1(z^D) + \cdots + H_{D-1}(z^D)
\end{aligned} \tag{4}$$

where,

$$H_k(z) = \sum_{n=-\infty}^{\infty} h(nD+k)z^{-n} \quad \text{for } 0 \le k < D \tag{5}$$

(c) If $H(z) = \frac{1}{1-az^{-1}}$ then

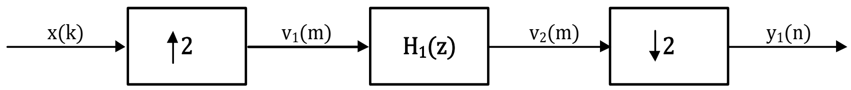
$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n} \tag{6}$$

Thus,

$$\begin{aligned}
H_0(z) &= \sum_{n=0}^{\infty} a^{2n} z^{-n} \\
&= \frac{1}{1-a^2z^{-1}}
\end{aligned} \tag{7}$$

$$\begin{aligned}
H_1(z) &= \sum_{n=0}^{\infty} a^{2n+1} z^{-n} \\
&= \frac{a}{1-a^2z^{-1}}
\end{aligned} \tag{8}$$

11.12



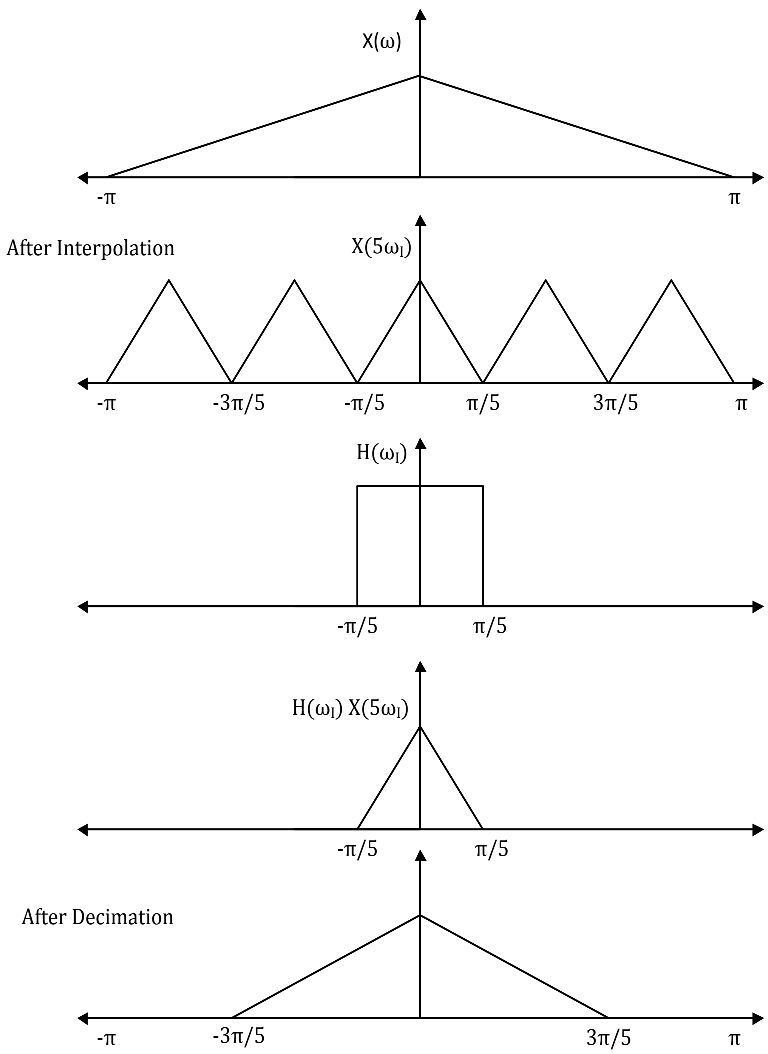
$$V_1(z) = X(z^2) \tag{1}$$

$$V_2(z) = H_1(z)V_1(z) \tag{2}$$

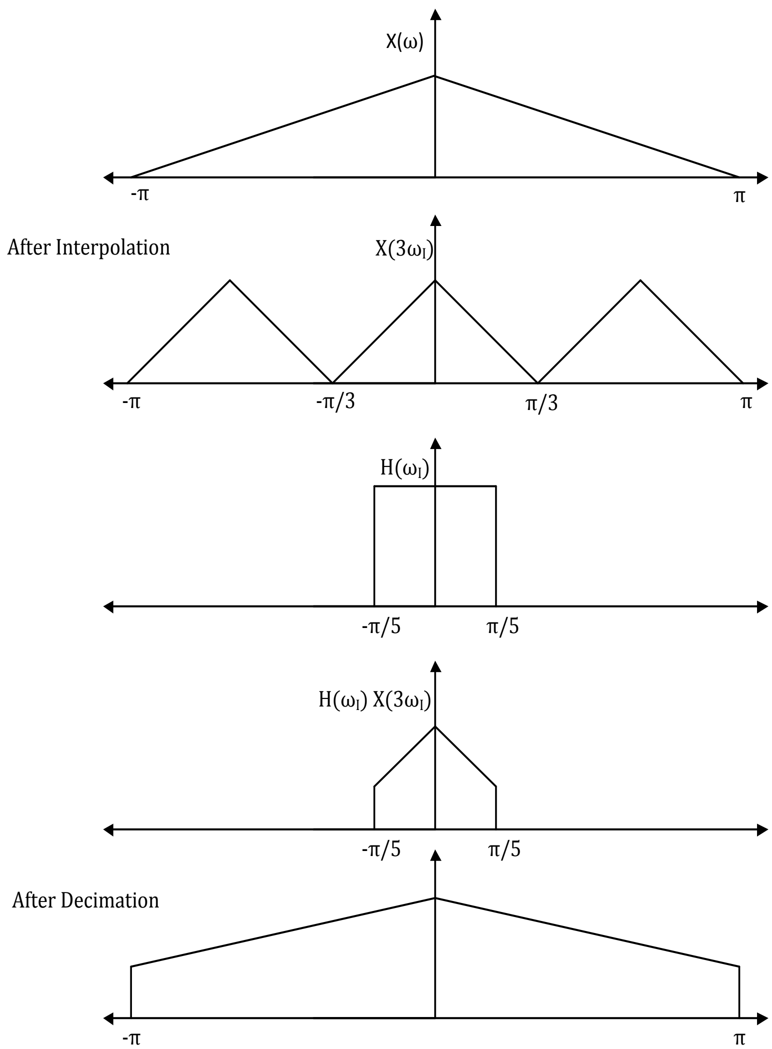
$$\begin{aligned}
Y(z) &= \frac{V_2(\sqrt{z}) + V_2(-\sqrt{z})}{2} && \text{(downsampler)} \\
&= \frac{H_1(\sqrt{z})V_1(\sqrt{z}) + H_1(-\sqrt{z})V_1(-\sqrt{z})}{2} && \text{(equation 2)} \\
&= \frac{H_1(\sqrt{z})X(z) + H_1(-\sqrt{z})X(z)}{2} && \text{(equation 1)} \\
&= \left(\frac{H_1(\sqrt{z}) + H_1(-\sqrt{z})}{2}\right) X(z) \\
&= H_2(z)X(z) && (3)
\end{aligned}$$

11.13

(a) Sketch for $I/D = 5/3$:



(b) Sketch for $I/D = 3/5$:



11.15

(a) The conventional polyphase decomposition is

$$H(z) = \sum_{m=0}^{N-1} z^{-m} P_m(z^N) \tag{1}$$

where $P_m(z) = \sum_{n=-\infty}^{\infty} h(nN + M) z^{-n}$.

This decomposition may be rewritten as

$$\begin{aligned} H(z) &= \sum_{m=0}^{N-1} z^{-m} P_m(z^N) \\ &= \sum_{m=0}^{N-1} z^{-(N-1-m)} P_{N-1-m}(z^N) \\ &= \sum_{m=0}^{N-1} z^{-(N-1-m)} Q_m(z^N) \end{aligned} \tag{2}$$

where,

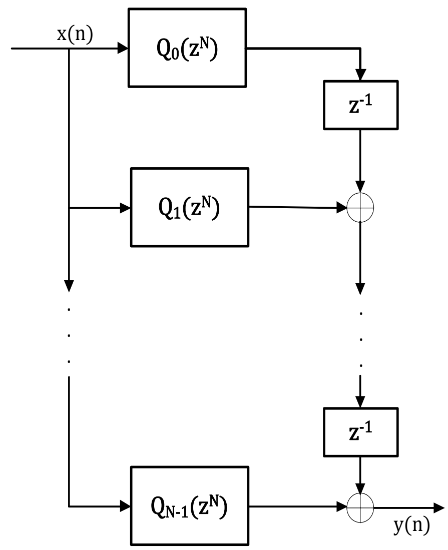
$$Q_m \triangleq P_{N-1-m}(z^N) \tag{3}$$

(b) If $X(z)$ is input to $H(z)$ then output

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= \sum_{m=0}^{N-1} z^{-(N-1-m)} X(z) Q_m(z^N) && \text{(equation 2)} \\ &= \sum_{m=0}^{N-1} V_m(z) Q_m(z^N) \end{aligned} \tag{4}$$

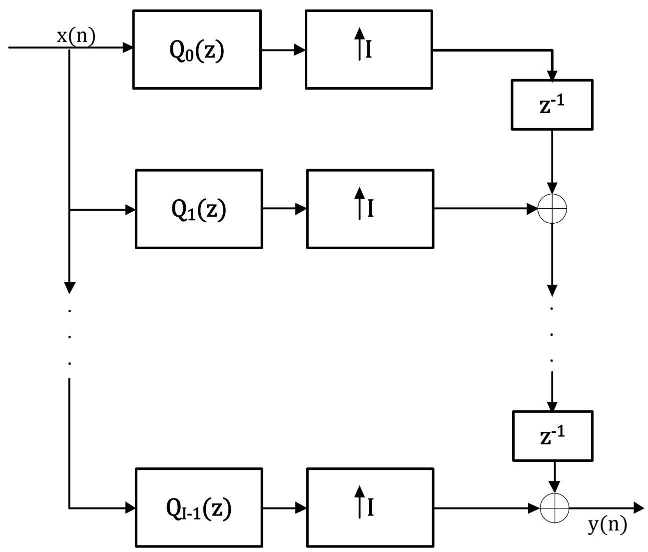
where, $V_m(z) \triangleq z^{-(N-1-m)} X(z)$.

This gives following structure



11.25

Type II polyphase decomposition of $H(z)$ (derived in problem 11.15) and noble identity imply realization:



11.29

$$H_0(z) = 1 + z^{-1} \tag{1}$$

(a)

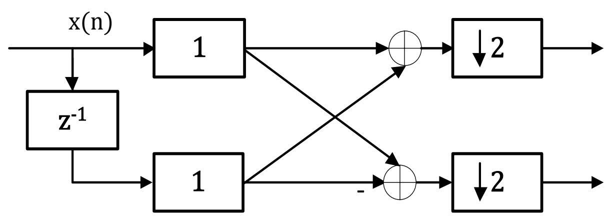
$$P_0(z) = \sum_n h_0(2n) z^{-n} = 1 \tag{2}$$

$$P_1(z) = \sum_n h_0(2n+1) z^{-(2n+1)} = 1 \tag{3}$$

(b)

$$\begin{aligned} H_1(z) &= P_0(z^2) - z^{-1} P_1(z^2) \\ &= 1 - z^{-1} \end{aligned} \tag{4}$$

Polyphase realization of analysis section is:

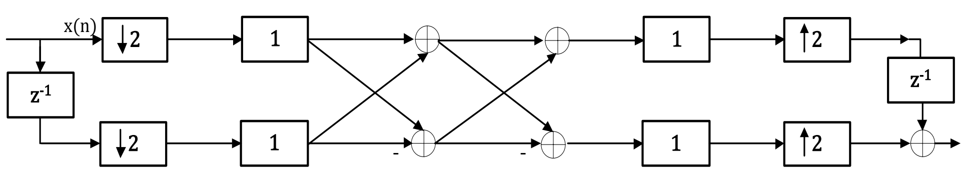


(c)

$$G_0(z) = P_0(z^2) + z^{-1}P_1(z^2)$$
$$= 1 + z^{-1} \quad (5)$$

$$G_1(z) = -\left(P_0(z^2) - z^{-1}P_1(z^2)\right)$$
$$= -1 + z^{-1} \quad (6)$$

A realization of two channel QMF based on polyphase filters is:



(d) For perfect reconstruction we must have

$$Q(z) = \frac{1}{2}\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right] = Cz^{-k} \quad (7)$$

We verify this requirement for this particular case:

$$Q(z) = \frac{1}{2}\left[(1+z^{-1})^2 - (1-z^{-1})^2\right]$$
$$= 2z^{-1} \quad (8)$$