http://www.comm.utoronto.ca/~dkundur/course/discrete-time-systems/

- Homework 8
Solutions

13.1

If plant output $d(n)$ is corrupted by white noise $w(n)$, let $y(n) = d(n) + w(n)$. Let output of the system model be

$$\hat{y}(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

The error is $e(n) = y(n) - \hat{y}(n)$ and it is required to minimize $\sum_{n=0}^{N} e^2(n)$. That is minimize,

$$\mathcal{E} = \sum_{n=0}^{N} \left(d(n) + w(n) - \sum_{k=0}^{M-1} h(k)x(n-k) \right)^2 \tag{1}$$

This can be done by differentiation (or methods of ecen601). The result is:

$$\sum_{k=0}^{M-1} h(k)r_{xx}(l-k) = r_{dx}(l) + r_{wx}(l), \quad l = 0, 1, \cdots, M-1 \tag{2}$$

where,

$$r_{xx}(l-k) = \sum_{n=0}^{N} x(n-k)x(n-l)$$

$$r_{wx}(l) = \sum_{n=0}^{N} w(n)x(n-l)$$

13.2

For a transmitted signal $s(t) = \sum_{k=0}^{\infty} a(k)p(t-kT_s)$, assuming presence of near end echo only, the received signal is (corrupted by white noise):

$$r(t) = As(t-d) + w(t)$$

The discrete time signal is (assuming sampling interval T):

$$r(n) = A \sum_{k=0}^{\infty} a(k) p(nT - kT - d) + w(nT)$$

The output of the echo canceller is $\hat{s}(n) = \sum_{k=0}^{M-1} h(k)a(n-k)$, and similar to problem 13.1, it is required to minimize

$$\mathcal{E} = \sum_{n=0}^{\infty} \left(r(n) - \sum_{k=0}^{M-1} h(k)a(n-k) \right)^2 \tag{1}$$

This minimization gives:

$$\sum_{k=0}^{M-1} h(k) r_{aa}(l-k) = r_{ra}(l), \quad l = 0, 1, \cdots, M-1 \tag{2}$$

where

$$r_{aa}(l-k) = \sum_{n=0}^{\infty} x(n-k)x(n-l)$$
$$r_{wx}(l) = \sum_{n=0}^{\infty} w(n)x(n-l)$$

13.3

$$\begin{aligned}
r_{vv}(k) &= \frac{1}{N} \sum_{n=0}^{N-1} v(n)v(n-k) \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \left(\sum_{l=0}^{\infty} h(l) w_2(n-l) + w_3(n) \right) \left(\sum_{m=0}^{\infty} h(m) w_2(n-k-m) + w_3(n-k) \right) \\
&= \frac{1}{N} \sum_{n=0}^{N=1} \left(\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} w_2(n-l) w_2(n-k-m) h(l) h(m) \right. \\
&\quad \left. + \sum_{l=0}^{\infty} w_2(n-l) w_3(n-k) h(l) + \sum_{m=0}^{\infty} w_2(n-k-m) w_3(n) h(m) + w_3(n) w_3(n-k) \right)
\end{aligned} \tag{1}$$

Since, $w_1(n), w_2(n), w_3(n)$ are mutually uncorrelated, $E[w_i(n-k)w_j(n-m)] = 0$ for $i \neq j$. Thus,

$$\begin{aligned}
E[r_{vv}(k)] &= \frac{1}{N} \sum_{n=0}^{N=1} \left(\sum_{l=0}^{\infty} \sum_{m=0}^{\infty} E[w_2(n-l) w_2(n-k-m)] h(l) h(m) + E[w_3(n) w_3(n-k)] \right) \\
&= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} E[w_2(n-l) w_2(n-k-m)] h(l) h(m) + E[w_3(n) w_3(n-k)]
\end{aligned} \tag{2}$$

$$
\begin{aligned}
r_{yv}(k) &= \frac{1}{N}\sum_{n=0}^{N-1} y(n)v(n-k) \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\Big(x(n)+w_2(n)+w_3(n)\Big)\left(\sum_{m=0}^{\infty} h(m)w_2(n-k-m)+w_3(n-k)\right) \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{m=0}^{\infty} h(m)x(n)w_2(n-k-m)+x(n)w_3(n-k)\right. \\
&\quad + \sum_{m=0}^{\infty} h(m)w_2(n)w_2(n-k-m)+w_2(n)w_3(n-k) \\
&\quad \left. + \sum_{m=0}^{\infty} h(m)w_2(n-k-m)w_3(n)+w_3(n)w_3(n-k)\right) \qquad (3)
\end{aligned}
$$

Since, $w_1(n), w_2(n), w_3(n)$ are mutually uncorrelated:

$$
\begin{aligned}
E[r_{yv}(k)] &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\sum_{m=0}^{\infty} h(m)E[x(n)w_2(n-k-m)]+E[x(n)w_3(n-k)]\right. \\
&\quad \left. + \sum_{m=0}^{\infty} h(m)E[w_2(n)w_2(n-k-m)]+E[w_3(n)w_3(n-k)]\right) \\
&= \sum_{m=0}^{\infty} h(m)E[x(n)w_2(n-k-m)]+E[x(n)w_3(n-k)] \\
&\quad + \sum_{m=0}^{\infty} h(m)E[w_2(n)w_2(n-k-m)]+E[w_3(n)w_3(n-k)] \qquad (4)
\end{aligned}
$$

13.12

$$
x(n) = gv(n) + w(n), \text{ for } 0 \le n < M \qquad (1)
$$

By orthogonality principle, $E[(g-\hat{g})^2]$ is minimized when $E[(g-\hat{g})x(k)] = 0$ for $0 \le k < M$.
We first evaluate $E[(g-\hat{g})x(k)]$:

$$
\begin{aligned}
E[(g-\hat{g})x(k)] &= E[gx(k)] - \sum_{n=0}^{M-1} h(n)E[x(n)x(k)] \\
&= E[g^2v(k)+gw(k)] - E[(gv(n)+w(n))(gv(k)+w(k))] \\
&= v(k)E[g^2] - v(n)v(k)E[g^2] - E[w(n)w(k)] \\
&= Gv(k) - Gv(n)v(k) - \delta(n-k)\sigma_w^2 \qquad (2)
\end{aligned}
$$

Thus, orthogonality principle implies

$$Gv(k) = \sum_{n=0}^{M-1} h(n)\left(v(n)v(k)G + \delta(n-k)\sigma_w^2\right) \quad \text{for } 0 \le k < M \tag{3}$$

Writing this set of equations in matrix form:

$$\left(\sigma_w^2 \mathbf{I} + G\mathbf{v}\mathbf{v}^T\right)\mathbf{h} = G\mathbf{v} \tag{4}$$

where

$$\mathbf{v} = [v(0)\cdots v(M-1)]^T \text{ and } \mathbf{h} = [h(0)\cdots h(M-1)]^T$$

Hence,

$$\mathbf{h} = G\left(\sigma_w^2 \mathbf{I} + G\mathbf{v}\mathbf{v}^T\right)^{-1}\mathbf{v} \tag{5}$$

Using matix inversion lemma we have

$$\mathbf{h} = \frac{1}{\sigma_w^2}G\mathbf{v} - \left(\frac{G\mathbf{v}\mathbf{v}^T}{\sigma_w^2(1 + G\mathbf{v}^T\mathbf{v})}\right)G\mathbf{v} \tag{6}$$

Simplifying we get

$$h(n) = \frac{Gv(n)}{\sigma_w^2\left(1 + G\sum_{k=0}^{M-1} v^2(k)\right)}, \quad \text{for } 0 \le n < M \tag{7}$$

13.15

Minimizing MSE, $E[e^2(n)]$, where $e(n) = y(n) - \sum_{k=0}^{M-1} a(k)v(n-k)$, gives the normal equations:

$$\sum_{k=0}^{M-1} a(k)r_{vv}(m-k) = r_{yv}(m), \quad \text{for } 0 \le m < M \tag{1}$$

We have to evaluate $r_{vv}(k)$ and $r_{yv}(k)$ for the particular case in this problem.

$$\begin{aligned} r_{vv}(k) &= E[v(n)v(n-k)] \\ &= E[v_2(n)v_2(n-k)] + E[w_3(n)w_3(n-k)] \\ &= r_{v_2v_2}(k) + r_{w_3w_3}(k) \\ &= r_{v_2v_2}(k) + \sigma_w^2\delta(k) \end{aligned} \tag{2}$$

To evaluate $r_{v_2v_2}(k)$, we first find power spectral density of $v_2(n)$ and take inverse DTFT.

$$\begin{aligned} S_{v_2v_2}(\omega) &= |H(\omega)|^2\sigma_w^2 \\ &= \frac{\sigma_w^2}{\left|1 - \frac{1}{2}e^{-jw}\right|^2} \end{aligned} \tag{3}$$

Inverse DTFT gives

$$r_{v_2 v_2}(k) = \frac{4\sigma_w^2}{3 \cdot 2^{|m|}} \tag{4}$$

Thus, equation 2 implies

$$r_{vv}(k) = \frac{4\sigma_w^2}{3 \cdot 2^{|m|}} + \sigma_w^2 \delta(k) \tag{5}$$

$$\begin{aligned}
r_{yv}(k) &= E[y(n)v(n-k)] \\
&= E[x(n)v_2(n-k)] + E[x(n)w_3(n-k)] + E[w_1(n)v_2(n-k)] + E[w_1(n)w_3(n-k)] \\
&\quad E[w_2(n)v_2(n-k)] = E[w_2(n)w_3(n-k)] \\
&= E[w_2(n)v_2(n-k) \qquad (6)
\end{aligned}$$

assuming $x(n), w_1(n), w_2(n)$ and $w_3(n)$ are uncorrelated. Hence,

$$\begin{aligned}
r_{yv}(k) &= \sum_{l=0}^{\infty} h(l) E[w_2(n)w_2(n-k-l)] \\
&= \sum_{l=0}^{\infty} h(l) \sigma_w^2 \delta(l+k) \\
&= \sigma_w^2 \delta(k) \qquad (7)
\end{aligned}$$

Use equations 5 and 7 to solve equation 1 for $M = 3$. This gives

$$a(0) = \frac{15}{32}, \quad a(1) = \frac{-1}{8}, \quad a(2) = \frac{-1}{32} \tag{8}$$