Multirate Digital Signal Processing: Part III

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Discrete-Time Signals and Systems

Reference:

Sections 11.5 and 11.9 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Polyphase Filter Structures

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

$$= \begin{cases} \cdots + & h(0) & + & h(M)z^{-M} & + \cdots & \text{(row 0)} \\ \cdots + & h(1)z^{-1} & + & h(M+1)z^{-(M+1)} & + \cdots & \text{(row 1)} \\ & \vdots & & \vdots & & \vdots \\ \cdots + & h(M-1)z^{-(M-1)} & + & h(2M-1)z^{-(2M-1)} & + \cdots & \text{(row } M-1) \end{cases}$$

$$= \begin{cases} z^{-0}[\cdots + & h(0) & + & h(M)z^{-M} & + \cdots] \\ z^{-1}[\cdots + & h(1) & + & h(M+1)z^{-M} & + \cdots] \\ & \vdots & & \vdots & \\ z^{-(M-1)}[\cdots + & h(M-1) & + & h(2M-1)z^{-M} & + \cdots] \end{cases}$$

$$= \sum_{i=0}^{M-1} z^{-i} \sum_{n=-\infty}^{\infty} h(nM+i)z^{-nM}$$

$$= \sum_{i=0}^{M-1} z^{-i} \sum_{n=-\infty}^{\infty} h(nM+i)(z^M)^{-n} = \sum_{i=0}^{M-1} z^{-i} P_i(z^M)$$

$$\underbrace{H(z) = \sum_{i=0}^{M-1} z^{-i} P_i(z^M)}_{M\text{-component polyphase decomposition}}$$

$$\underbrace{P_i(z^M) = \sum_{n=-\infty}^{\infty} h(nM+i)z^{-nM} = \sum_{n=-\infty}^{\infty} p_i(n)z^{-nM}}_{\text{Polyphase components of } H(z)}$$

Observe:

$$p_i(n) = h(nM+i), \quad i = 0, 1, 2, \ldots, M-1$$

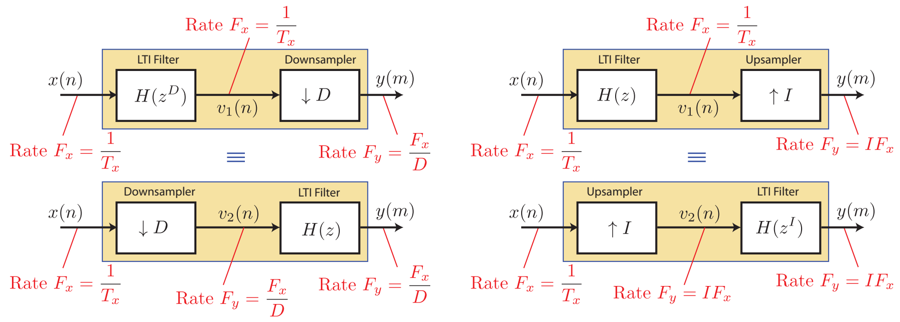
which is a downsampled and delayed ("phase shifted") version of the original impulse response.

Consider LTI filtering of an input $x(n)$ with filter $H(z)$ using a polyphase filter structure with $M = 3$.

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \left[\sum_{i=0}^{2} z^{-i} P_i(z^M)\right] X(z) \\ &= [P_0(z^3) + z^{-1}P_1(z^3) + z^{-2}P_2(z^3)]X(z) \\ &= P_0(z^3)X(z) + z^{-1}P_1(z^3)X(z) + z^{-2}P_2(z^3)X(z) \\ &= P_0(z^3)X(z) + z^{-1}\{P_1(z^3)X(z) + z^{-1}[P_2(z^3)X(z)]\} \end{aligned}$$

See ‣ Figure 11.5.1 of text.
See ‣ Figure 11.5.2 of text.

Noble Identities

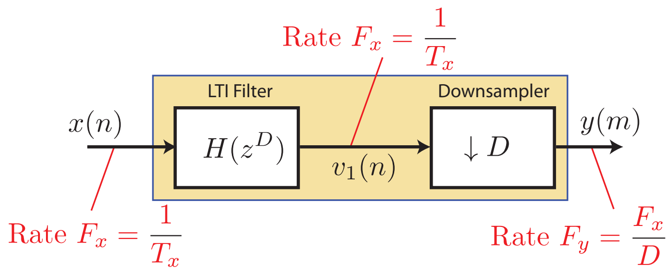


Recall, for a downsampler:

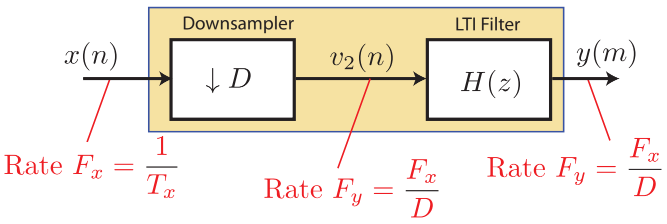
$$v(n) = u(nD) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad V(z) = \frac{1}{D}\sum_{i=0}^{D-1} U(z^{1/D}W_D^i)$$

where $W_D = e^{-j2\pi/D}$.

$$v(n) = u(nD) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad V(z) = \frac{1}{D}\sum_{i=0}^{D-1} U\left(\frac{\omega - 2\pi i}{D}\right)$$

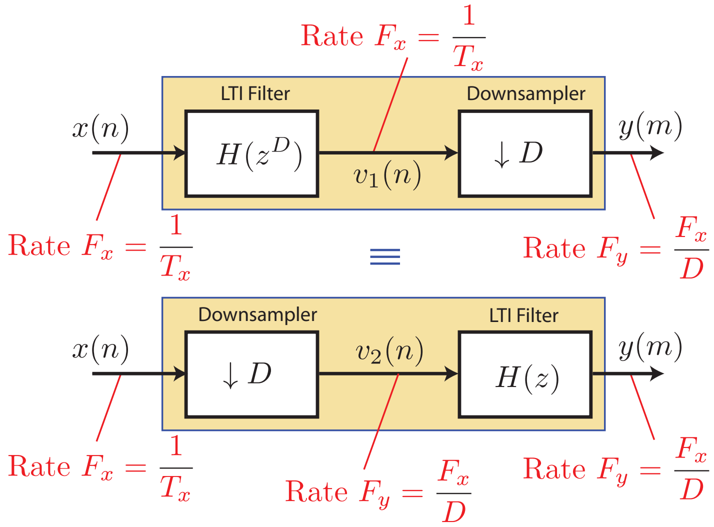


$$\begin{aligned} V_1(z) &= H(z^D)X(z) \\ Y(z) &= \frac{1}{D}\sum_{i=0}^{D-1} V_1(z^{1/D}W_D^i) \\ &= \frac{1}{D}\sum_{i=0}^{D-1} H((z^{1/D}W_D^i)^D)X(z^{1/D}W_D^i) \\ &= \frac{1}{D}\sum_{i=0}^{D-1} H(z \underbrace{W_D^{iD}}_{=e^{-j2\pi D/D}})X(z^{1/D}W_D^i) = \boxed{\frac{1}{D}\sum_{i=0}^{D-1} H(z)X(z^{1/D}W_D^i)} \end{aligned}$$



$$\begin{aligned} V_2(z) &= \frac{1}{D}\sum_{i=0}^{D-1} X(z^{1/D}W_D^i) \\ Y(z) &= H(z)V_2(z) \\ &= H(z)\frac{1}{D}\sum_{i=0}^{D-1} X(z^{1/D}W_D^i) \\ &= \boxed{\frac{1}{D}\sum_{i=0}^{D-1} H(z)X(z^{1/D}W_D^i)} \end{aligned}$$

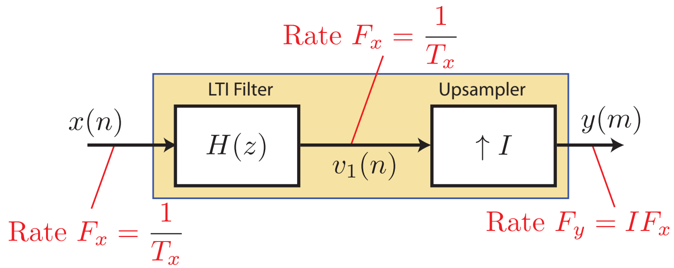
Noble Identity – Decimation



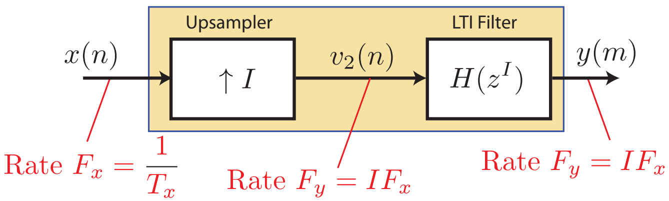
Recall, for an upsampler:

$$v(n) = \begin{cases} u\left(\frac{n}{I}\right) & n = 0, \pm I, \pm 2I, \ldots \\ 0 & \text{otherwise} \end{cases} \quad \overset{z}{\longleftrightarrow} \quad V(z) = U(z^I)$$

$$v(n) = \begin{cases} u\left(\frac{n}{I}\right) & n = 0, \pm I, \pm 2I, \ldots \\ 0 & \text{otherwise} \end{cases} \quad \overset{z}{\longleftrightarrow} \quad V(\omega) = U(\omega I)$$

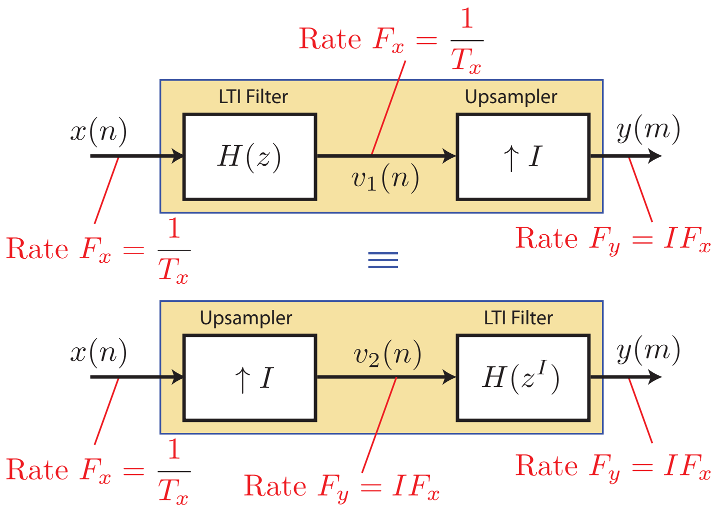


$$\begin{aligned} V_1(z) &= H(z)X(z) \\ Y(z) &= V_1(z^I) = \boxed{H(z^I)X(z^I)} \end{aligned}$$



$$
\begin{aligned}
V_2(z) &= X(z^I) \\
Y(z) &= H(z^I)V_2(z) = \boxed{H(z^I)X(z^I)}
\end{aligned}
$$

Noble Identity – Interpolation

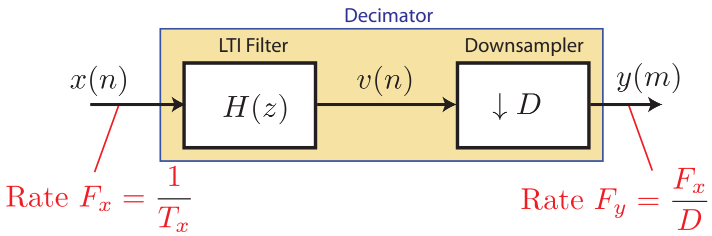


Noble Identities

It is possible to interchange the operation of LTI filtering and downsampling or upsampling if we properly modify the system function of the filter.

Polyphase Structures of Decimation Filters

Consider



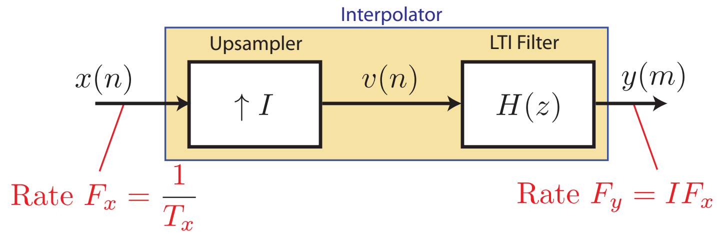
Consider a polyphase implementation with $M = 3$.
See ▸ Figure 11.5.9 of text.

Polyphase Structures of Decimation Filters

- Use of the Noble identity allows <u>reduction</u> of number of multiplications and additions, since filtering is performed at a lower rate.
- It is more convenient to implement the polyphase decimator using a <u>commutator model</u>.
  See ▸ Figure 11.5.10 of text.

Polyphase Structures of Interpolation Filters

Consider



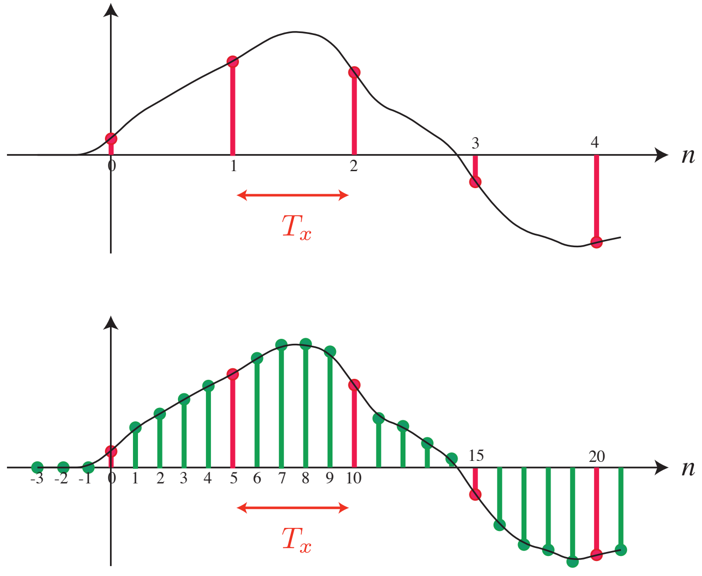
Consider a transpose polyphase implementation with $M = 3$.
See ▸ Figure 11.5.12 of text.

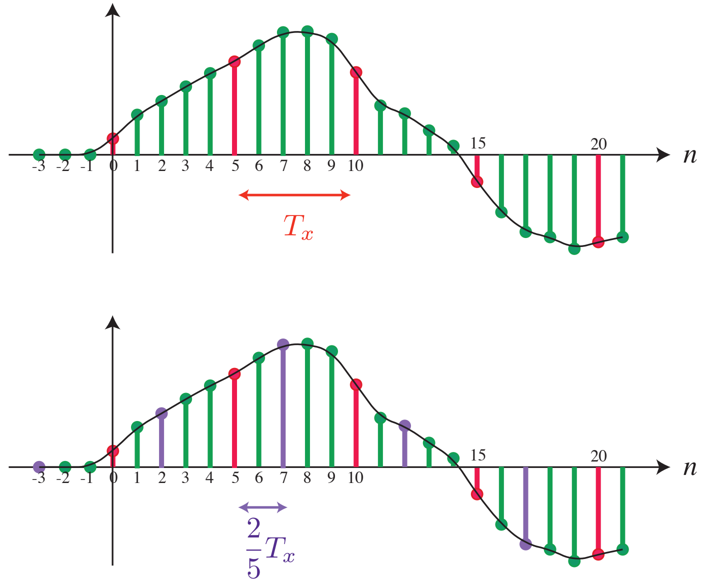
Polyphase Structures of Interpolation Filters

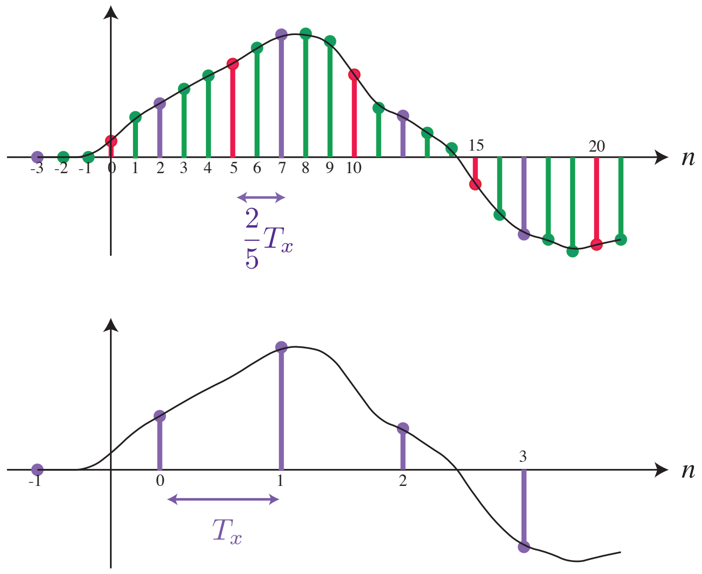
- Use of the Noble identity allows <u>reduction</u> of number of multiplications and additions, since filtering is performed at a lower rate.
- It is more convenient to implement the polyphase decimator using a <u>commutator model</u>.
  See ▸ Figure 11.5.13 of text.

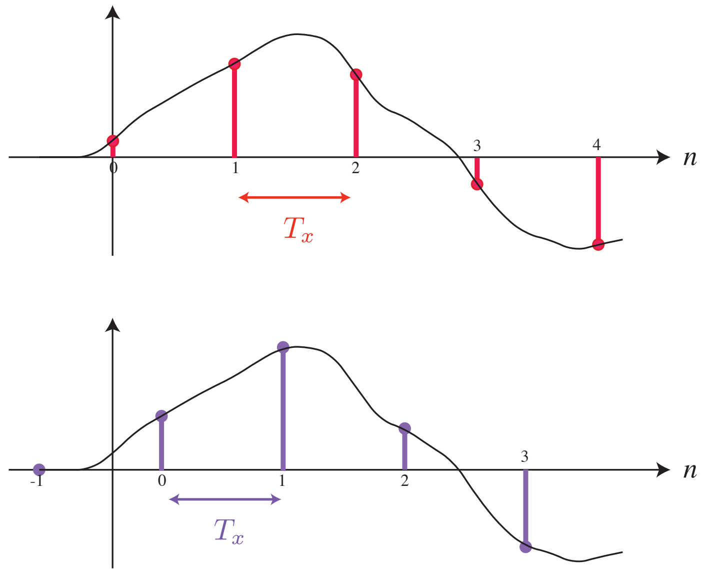
Phase Shifter

- Phase shifter: system that delays a signal $x(n)$ by a fraction of a sample.
- Consider a delay that is a rational fraction of a sampling interval.
- Example: $\frac{2}{5}T_x$ delay

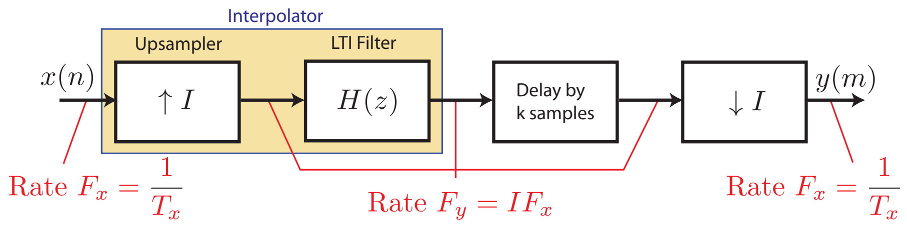

0 1 2 3 4
n
T_x
-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 15 20
n
T_x


-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 15 20
n
T_x
-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 15 20
n
$\frac{2}{5}T_x$


-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 15 20
n
$\frac{2}{5}T_x$
-1 0 1 2 3
n
T_x


0 1 2 3 4
n
T_x
-1 0 1 2 3
n
T_x

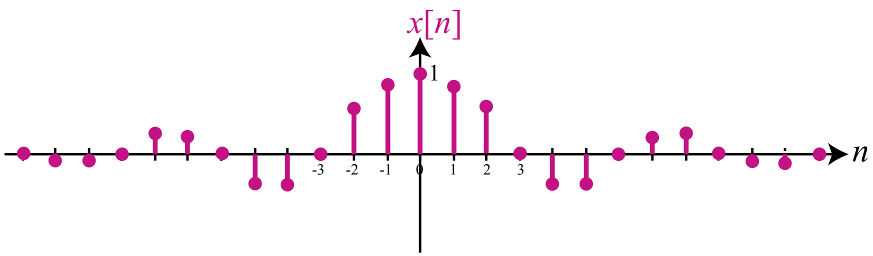
Fractional Phase Shifter

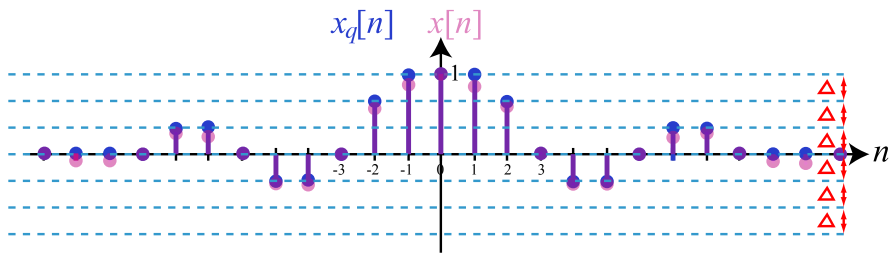


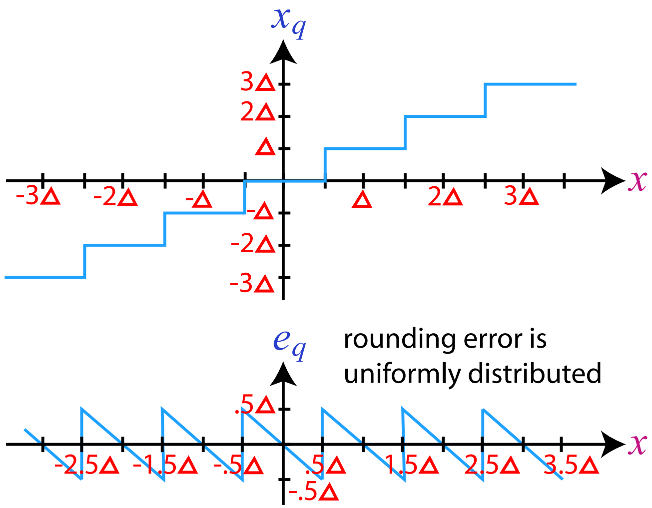
- Efficient implementation makes use of polyphase filter structures for the $H(z)$ filter and a commutator implementation. See ‣ Figure 11.9.2 of text.
- In particular, fixing the commutator location provides the desired delay and downsampling.
- The implementation allows for shifts of $\{0, 1/I, 2/I, \ldots, (I-1)/I\}$ depending on the fixed position of the commutator.

Subband Coding of Speech Signals

- Goal: efficiently represent speech signals in digital form.
- Characteristic: most speech energy is contained in lower frequencies.
- Idea: encode higher-frequency bands with fewer bits/sample than lower-frequency bands.
  - bits/sample is related to the amplitude quantization level
  - lower number of bits/sample implies greater degree of amplitude quantization



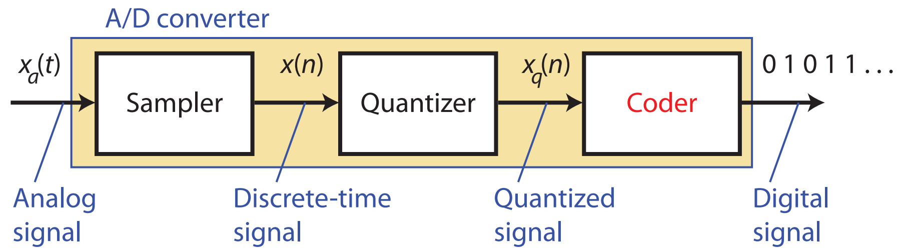




- larger Δ results in higher quantization error

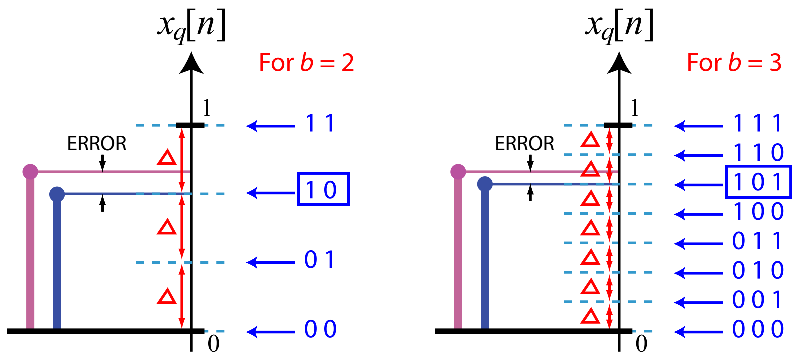
Coding

- larger Δ results in $x_q(n)$ that requires fewer bits/sample to represent.



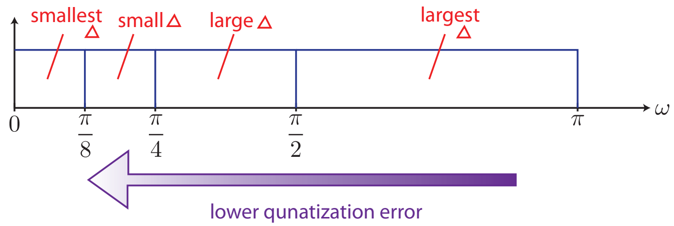
Coding

- larger Δ results in $x_q(n)$ that requires fewer bits/sample to represent.



Coding

smaller Δ $\Longleftrightarrow$ $\begin{cases} \text{smaller quantization error} \\ \text{greater number of quantization levels} \\ \text{larger bits/sample representation} \end{cases}$



- lower quantization error occurs at lower frequencies where significant speech signal energy exists
- larger degree of quantization allows more efficient coding

Multirate Implementation of Subband Coder

Recall, for a downsampler and upsampler:

$$v(n) = u(nD) \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \underbrace{V(z) = \frac{1}{D}\sum_{i=0}^{D-1} U(z^{1/D}e^{-j2\pi i/D})}_{\text{BW } \boxed{\text{expansion}} \text{ by factor } D}$$

$$v(n) = \begin{cases} u\left(\frac{n}{I}\right) & n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \underbrace{V(z) = U(z^I)}_{\text{BW } \boxed{\text{compression}} \text{ by factor } I}$$

See ▸ Figure 11.9.4 of text.

See ▸ Figure 11.9.6 of text. ■