Multirate Digital Signal Processing: Part IV

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Discrete-Time Signals and Systems

Reference:

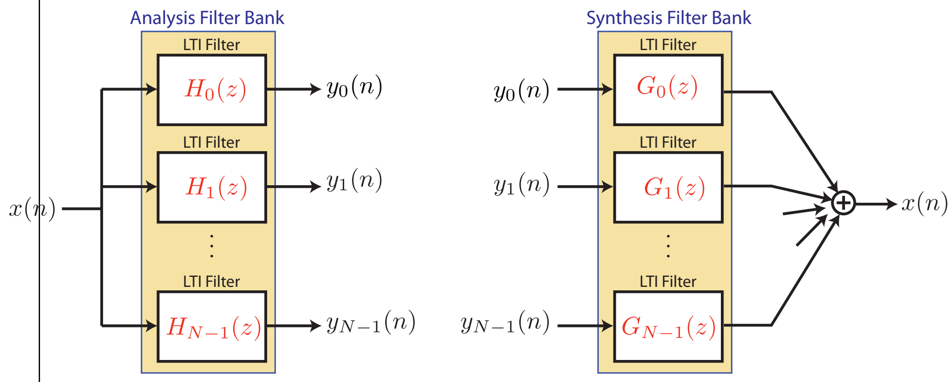
Sections 11.10 and 11.11 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Filter Banks

- Two types: analysis and synthesis
- consist of a parallel <u>bank</u> of filters used for:
  - signal analysis, DFT computation, etc.
  - signal (re-)synthesis

Analysis and Synthesis Filter Banks



Analysis Filter Bank

Consider Uniform DFT Filter Bank

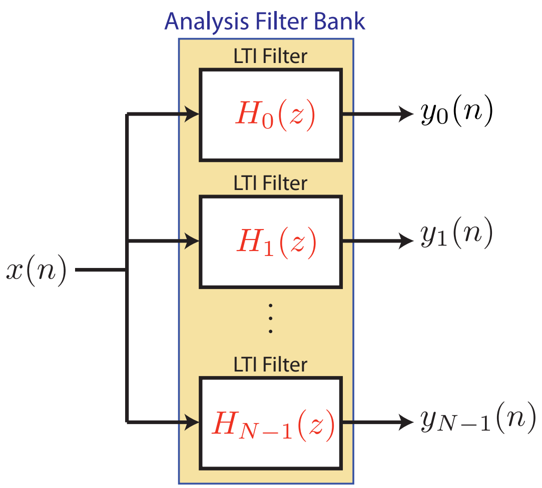
1. analysis filter bank
2. N filters $\{H_k(z), k = 0, 1, \ldots, N-1\}$
3. prototype filter: $H_0(z)$

$$
\begin{aligned}
H_k(z) &= H_0\left(z e^{-j2\pi k/N}\right) \\
H_k(e^{j2\pi\omega}) &= H_0\left(e^{j\omega} e^{-j2\pi k/N}\right) \\
H_k(\omega) &= H_0\left(\omega - \frac{2\pi k}{N}\right) \\
&= H_0(\omega) * \delta\left(\omega - \frac{2\pi k}{N}\right)
\end{aligned}
$$

Prototype Filter

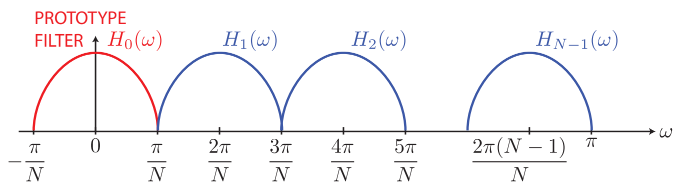
For $k = 0, 1, 2, \ldots, N$:

$$
\begin{aligned}
H_k(\omega) &= H_0\left(\omega - \frac{2\pi k}{N}\right) \\
\boxed{h_k(n)} &\overset{\mathcal{F}}{\longleftrightarrow} H_k(\omega) \\
h_0(n) &\overset{\mathcal{F}}{\longleftrightarrow} H_0(\omega) \\
\boxed{h_0(n) e^{j2\pi nk/N}} &\overset{\mathcal{F}}{\longleftrightarrow} H_0\left(\omega - \frac{2\pi k}{N}\right) \\
\therefore \quad h_k(n) &= h_0(n) e^{j2\pi nk/N}
\end{aligned}
$$



Note: $h_k(n) = h_0(n) e^{j2\pi nk/N} \overset{\mathcal{Z}}{\longleftrightarrow} H_k(z) = H_0\left(z e^{-j2\pi k/N}\right)$

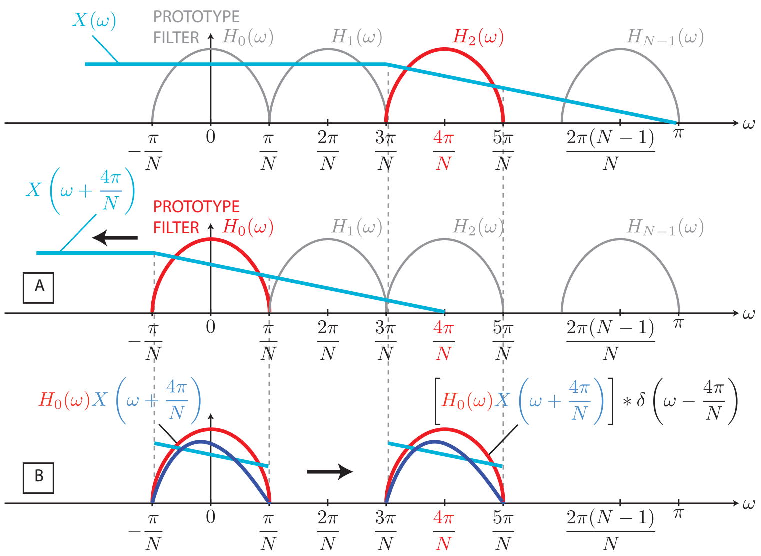
Uniform DFT Filter Bank



$$
\begin{aligned}
h_0(n) &\overset{\mathcal{F}}{\longleftrightarrow} H_0(\omega) \\
x(n) &\overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \\
e^{+j\frac{2\pi kn}{N}} &\overset{\mathcal{F}}{\longleftrightarrow} \delta\left(\omega - \frac{2\pi kn}{N}\right) \\
e^{-j\frac{2\pi kn}{N}} &\overset{\mathcal{F}}{\longleftrightarrow} \delta\left(\omega + \frac{2\pi kn}{N}\right)
\end{aligned}
$$

$$
h_k(n) = h_0(n) \cdot e^{j\frac{2\pi kn}{N}} \overset{\mathcal{F}}{\longleftrightarrow} H_k(\omega) = H_0(\omega) * \delta\left(\omega - \frac{2\pi k}{N}\right)
$$

Uniform DFT Filter Bank



Therefore,

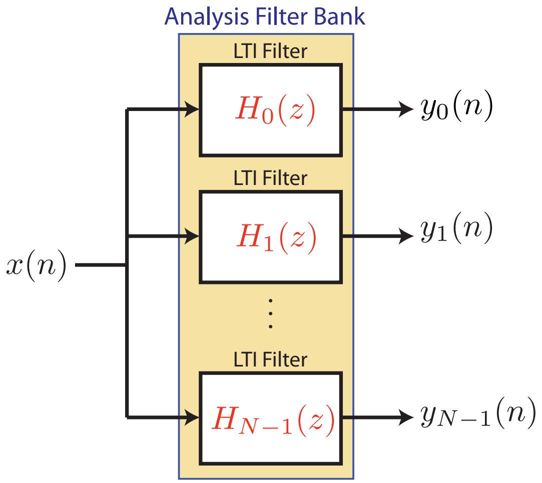
$$Y_k(\omega) = H_k(\omega)\cdot X(\omega)$$

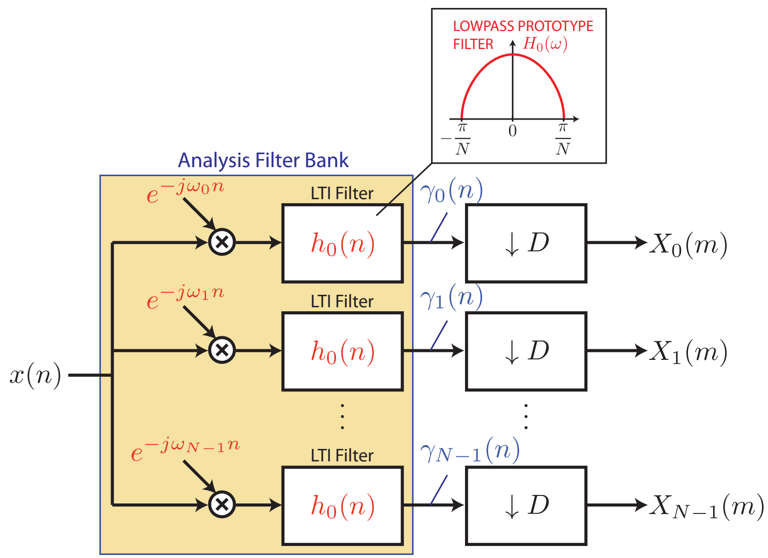
$$= \underbrace{\overbrace{\left[H_0(\omega)\cdot X\left(\omega+\frac{2\pi k}{N}\right)\right]}^{\boxed{A}} * \delta\left(\omega-\frac{2\pi k}{N}\right)}_{\boxed{B}}$$

$$= \left\{ H_0(\omega)\cdot \underbrace{\left[X(\omega)*\delta\left(\omega+\frac{2\pi k}{N}\right)\right]}_{\text{compensate at analysis bank}} \right\} \overbrace{*\ \delta\left(\omega-\frac{2\pi k}{N}\right)}^{\text{compen. at synthesis bank}}$$

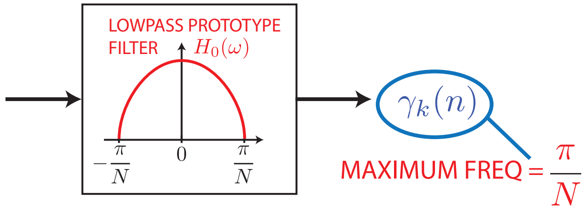
$$x(n)\cdot \underbrace{e^{-j2\pi kn/N}}_{=e^{-j\omega_k n}} \overset{\mathcal{F}}{\longleftrightarrow} \left[X(\omega)*\delta\left(\omega+\frac{2\pi k}{N}\right)\right]$$

where $\boxed{\omega_k = 2\pi kn/N}$.



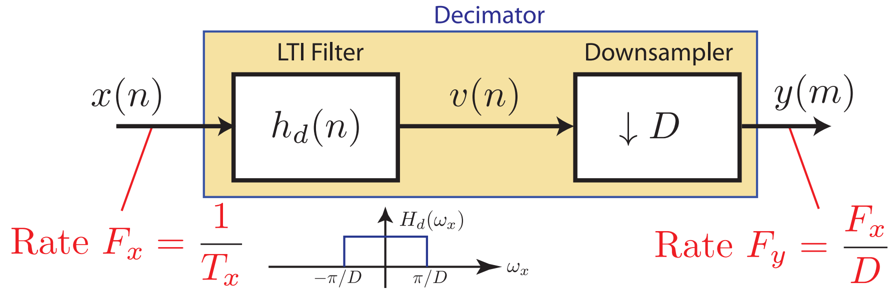


Note:
downsampling reduces redundancy without loss.



- $\gamma_k(n)$ is bandlimited such that it is oversampled by a factor of $N \gg 1$
- $H_0(\omega)$ behaves as a anti-aliasing filter prior to decimation.
- Recall, ...

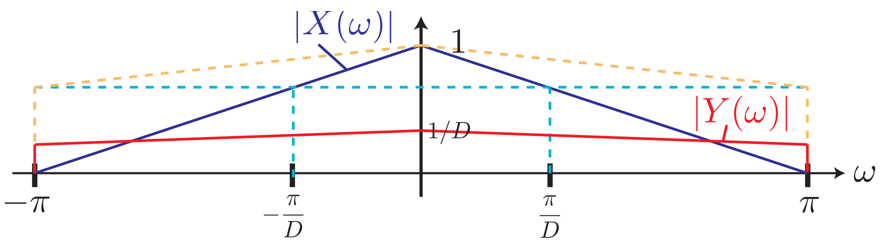
Recall,



$$Y(\omega_y) = \frac{1}{D} X\left(\frac{\omega_y}{D}\right)$$

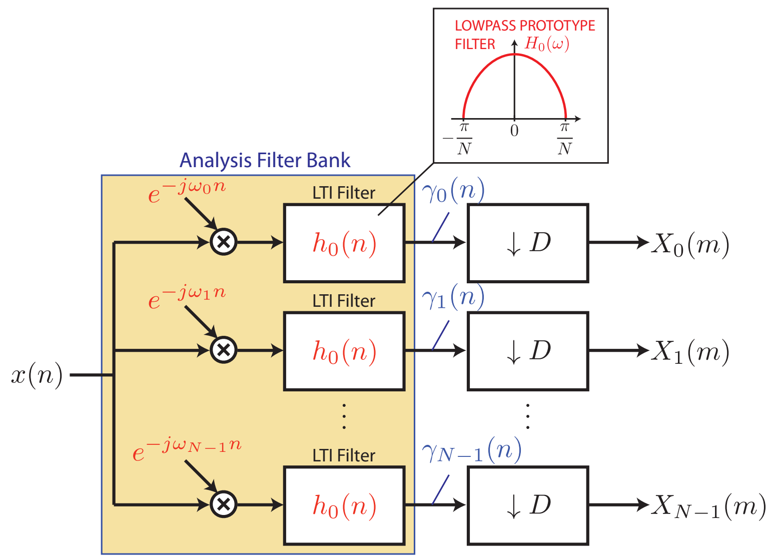
Note: $y(m)$ contains the same information as $v(m)$; thus downsampling of $v(m)$ is lossless compression.

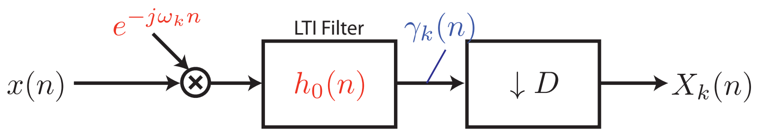
Recall for $D = 3$,



$$Y(\omega_y) = \frac{1}{D} X\left(\frac{\omega_y}{D}\right)$$

$-\frac{\pi}{D} \leq \omega_x \leq \frac{\pi}{D}$ of $X(\omega_x)$ is stretched into $-\pi \leq \omega_y \leq \pi$ for $Y(\omega_y)$.

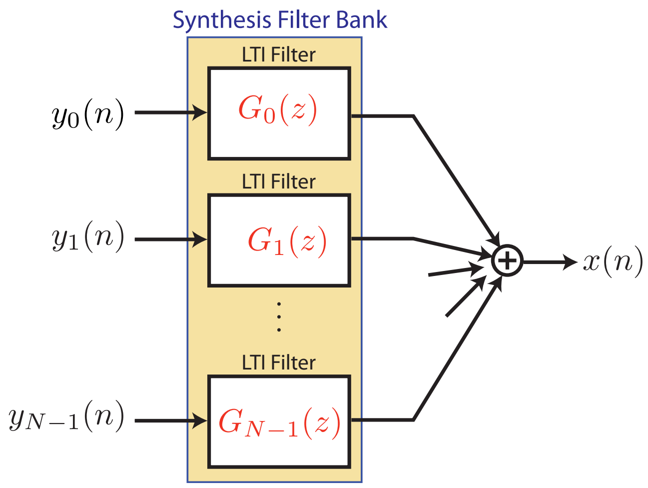




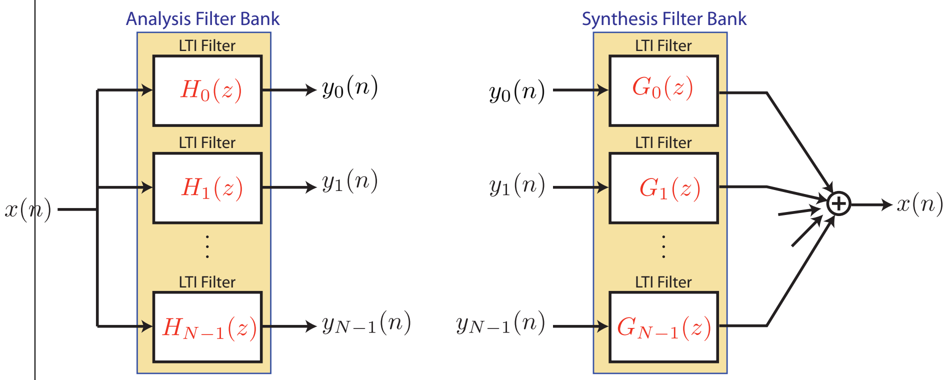
$$\begin{aligned}
X_k(m) &= \gamma_k(mD) \\
\gamma_k(n) &= [x(n) \cdot e^{-j\omega_k n}] * h_0(n) \\
&= \sum_{l=-\infty}^{\infty} x(l) \cdot e^{-j\omega_k l} h_0(n-l) \\
&= \sum_{l=-\infty}^{\infty} h_0(n-l) x(l) \cdot e^{-j\omega_k l} \\
X_k(m) &= \gamma_k(mD) = \sum_{n=-\infty}^{\infty} h_0(mD-n)x(n) \cdot e^{-j2\pi nk/N}
\end{aligned}$$

$$\boxed{X_k(m) = \sum_{n=-\infty}^{\infty} h_0(mD-n)x(n)e^{-j2\pi nk/N} \equiv k\text{th DFT coeff}}$$

Synthesis Filter Banks

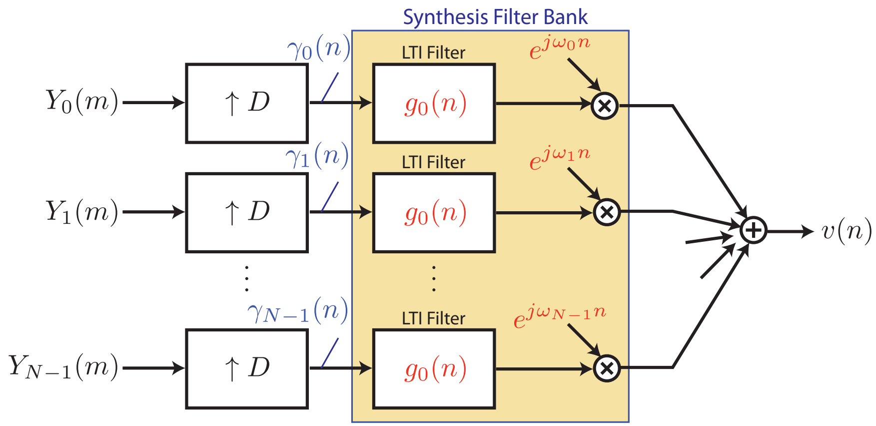


Analogous structure to analysis filter bank:



Application-based processing may occur between the analysis and synthesis filter banks.

Analogous structure to analysis filter bank:



Note: compensation for baseband processing in analysis bank.

Therefore,

$$Y_k(\omega) = H_k(\omega)\cdot X(\omega)$$

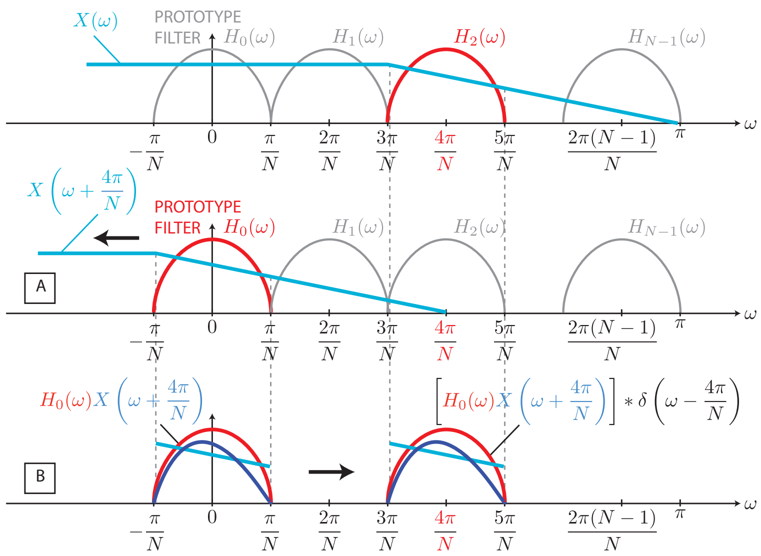
$$= \overbrace{\underbrace{\left[H_0(\omega)\cdot X\left(\omega+\frac{2\pi k}{N}\right)\right]}_{}}^{\boxed{A}} * \delta\left(\omega-\frac{2\pi k}{N}\right)\ \ \boxed{B}$$

$$= \left\{H_0(\omega)\cdot \underbrace{\left[X(\omega)*\delta\left(\omega+\frac{2\pi k}{N}\right)\right]}_{\text{compensate at analysis bank}}\right\} \overbrace{* \quad \delta\left(\omega-\frac{2\pi k}{N}\right)}^{\text{compen. at synthesis bank}}$$

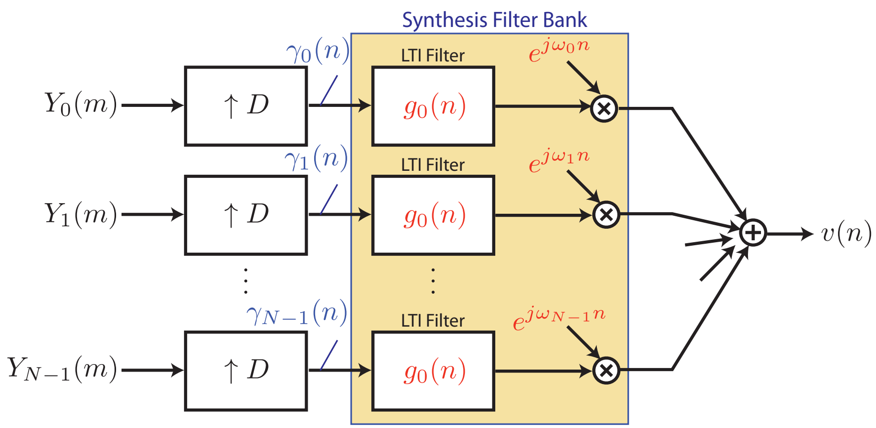
$$r(n)\cdot \underbrace{e^{+j2\pi kn/N}}_{=e^{j\omega_k n}} \overset{\mathcal{F}}{\longleftrightarrow} \left[R(\omega)*\delta\left(\omega-\frac{2\pi k}{N}\right)\right]$$

where $\boxed{\omega_k = 2\pi kn/N}$.

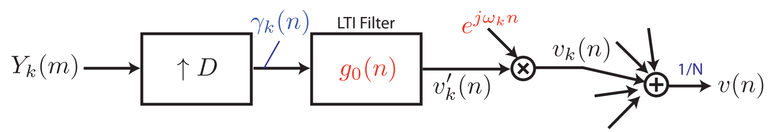
Uniform DFT Filter Bank



Analogous structure to analysis filter bank:



Note: compensation for baseband processing in analysis bank.



$$\gamma_k(n) = \begin{cases} Y_k(n/D) & n = 0, \pm D, \pm 2D, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$v_k'(n) = g_0(n)*\gamma_k(n) = \sum_{m=-\infty}^{\infty} \gamma_k(m) g_0(n-m)$$

$$= \cdots + \gamma_k(-D)g_0(n+D) + \gamma_k(0)g_0(n) + \gamma_k(D)g_0(n-D) + \cdots$$

$$= \sum_{m=-\infty}^{\infty} \gamma_k(mD) g_0(n-mD) = \sum_{m=-\infty}^{\infty} Y_k(m) g_0(n-mD)$$

$$v(n) = \frac{1}{N}\sum_{k=0}^{N-1} v_k(n) = \frac{1}{N}\sum_{k=0}^{N-1} v_k'(n) e^{j2\pi nk/N}$$

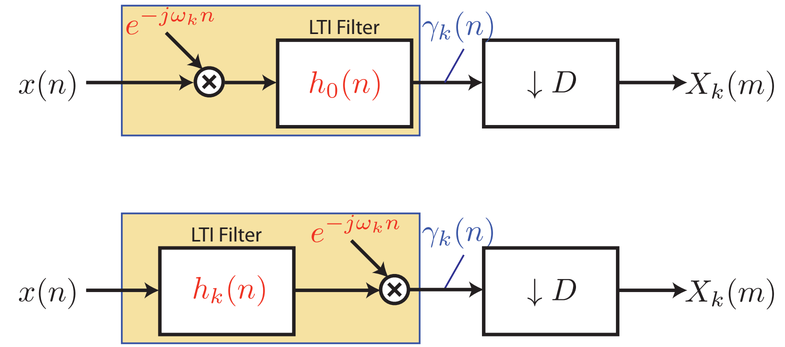
$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} Y_k(m) g_0(n-mD) e^{j2\pi nk/N}$$

$$v(n) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} Y_k(m) g_0(n-mD) e^{j2\pi nk/N}$$

$$= \sum_{m=-\infty}^{\infty} \underbrace{g_0(n-mD)}_{\text{interpol. fn}} \underbrace{\left[\frac{1}{N}\sum_{k=0}^{N-1} Y_k(m) e^{j2\pi nk/N}\right]}_{\equiv y_n(m) \equiv \text{IDFT coeff}}$$

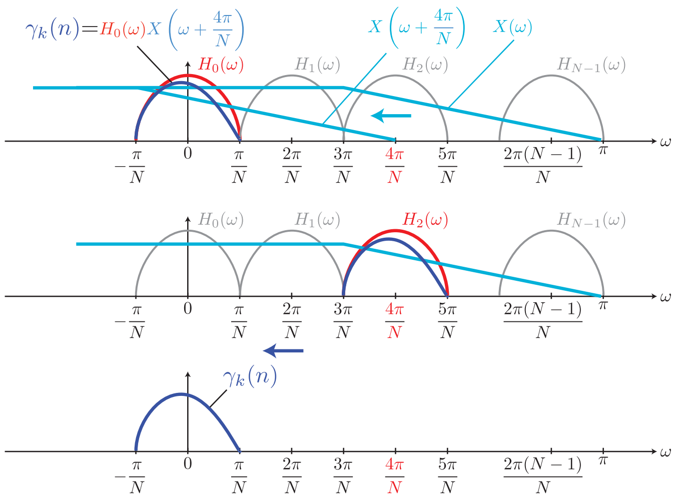
$$v(m) = \sum_{m=-\infty}^{\infty} g_0(n-mD)\left[\frac{1}{N}\sum_{k=0}^{N-1} Y_k(m) e^{j2\pi nk/N}\right] \equiv \text{IDFT}$$

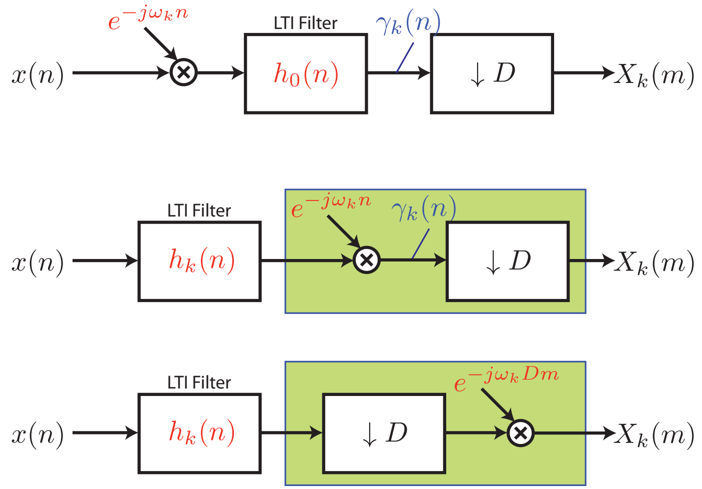
Alternative Filter Bank Structures



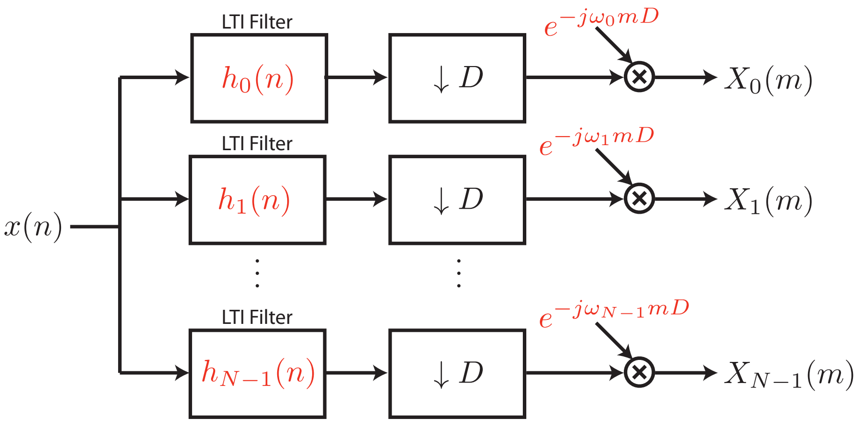
$$h_k(n) = h_0(n) e^{j2\pi nk/N}$$

$$H_k(\omega) = H_0\left(\omega - \frac{2\pi nk}{N}\right)$$



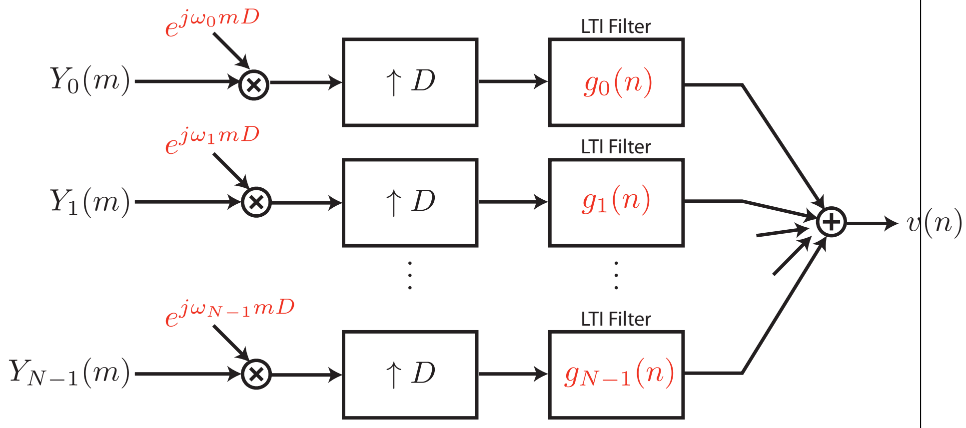


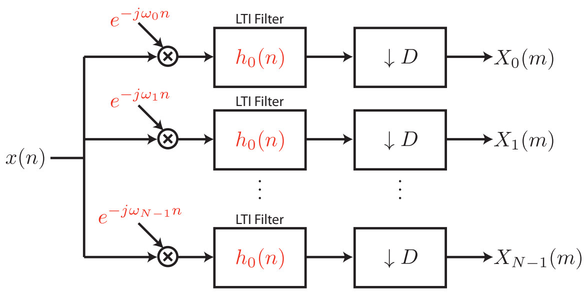
Alternative Filter Bank Structures



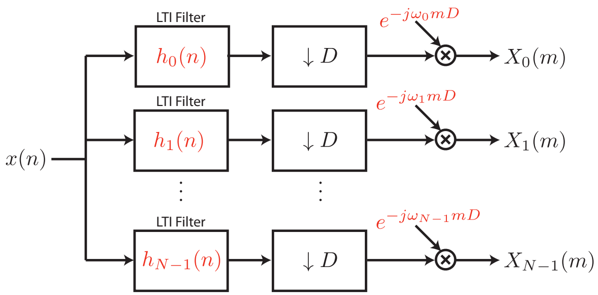
Alternative Filter Bank Structures

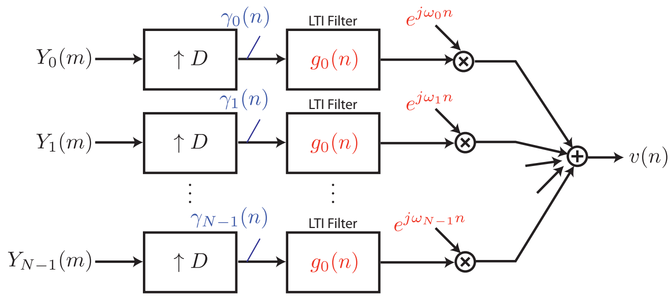
Similarly, the synthesis filter bank is given by:



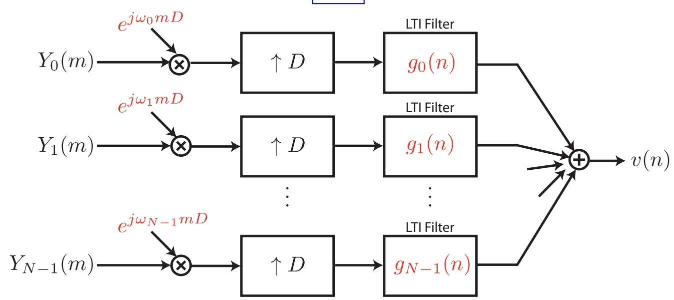


vs.





vs.



Critically Sampled Filter Banks

$$\boxed{D = N}$$

- maximizes efficiency by minimizing the number of samples in computations

Two-Channel Quadrature Mirror Filter Bank

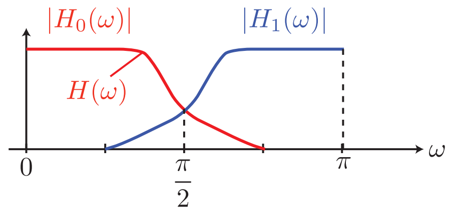
- multirate digital filter structure that employs:
  - two decimators for "signal analysis"
  - two interpolators "signal synthesis"
- basic building block for quadrature mirror filter (QMF) applications

Two-Channel Quadrature Mirror Filter Bank

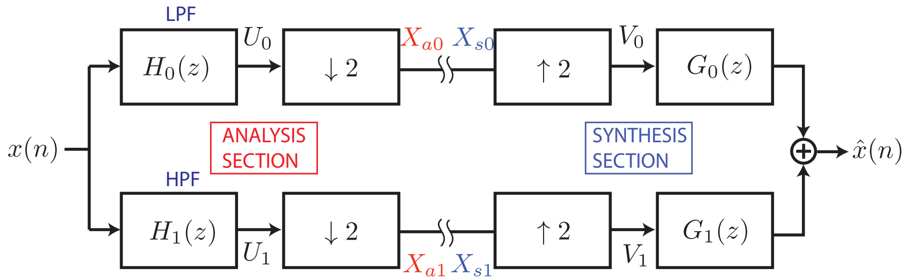
Q: What is a quadrature mirror filter?

A: Consider analysis filters $H_0(\omega)$ and $H_1(\omega)$ that are lowpass and highpass, respectively.

$$\begin{aligned} H_0(\omega) &= H(\omega) \\ H_1(\omega) &= H(\omega - \pi) \end{aligned}$$



Two-Channel Quadrature Mirror Filter Bank



Recall, for a downsampler:

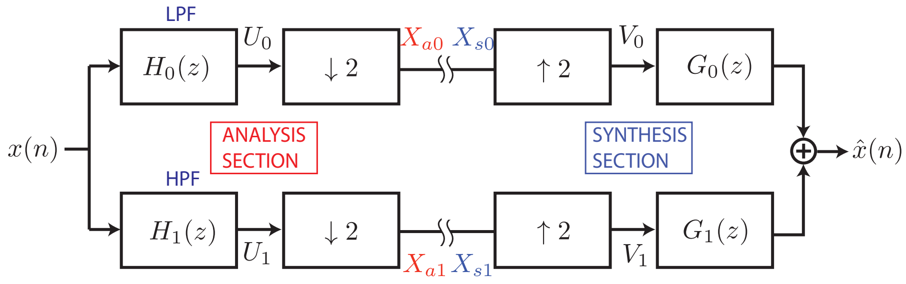
$$x(n) = u(nD) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad X(z) = \frac{1}{D}\sum_{i=0}^{D-1} U(z^{1/D}W_D^i)$$

where $W_D = e^{-j2\pi/D}$.

$$x(n) = u(nD) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad X(z) = \frac{1}{D}\sum_{i=0}^{D-1} U\left(\frac{\omega - 2\pi i}{D}\right)$$

For $D = 2$:

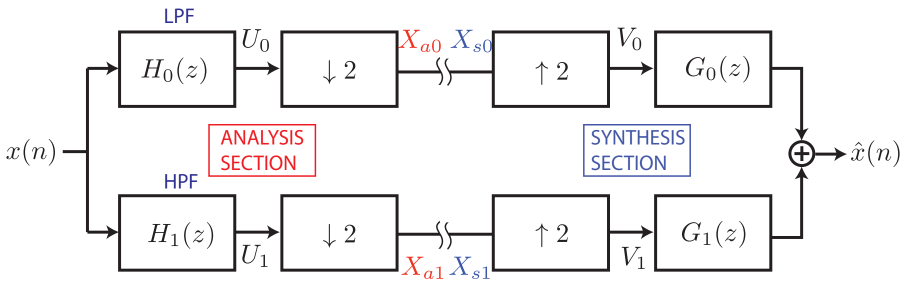
$$\begin{aligned} X(z) &= \frac{1}{2}\sum_{i=0}^{2-1} U\left(\frac{\omega - 2\pi i}{2}\right) \\ &= \frac{1}{2}\left[U\left(\frac{\omega}{2}\right) + U\left(\frac{\omega - 2\pi}{2}\right)\right] \end{aligned}$$



$$X_{a0}(\omega) = \frac{1}{2}\left[U_0\left(\frac{\omega}{2}\right) + U_0\left(\frac{\omega - 2\pi}{2}\right)\right]$$

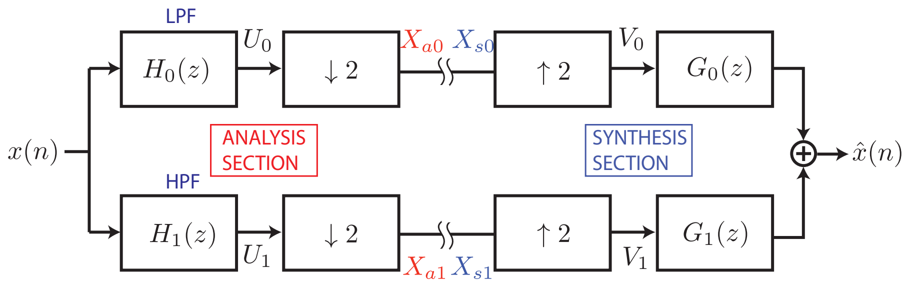
Note: $U_0(\omega) = H_0(\omega)X(\omega)$

$$\begin{aligned} X_{a0}(\omega) &= \frac{1}{2}\left[H_0\left(\frac{\omega}{2}\right)X\left(\frac{\omega}{2}\right) + H_0\left(\frac{\omega - 2\pi}{2}\right)X\left(\frac{\omega - 2\pi}{2}\right)\right] \\ X_{a1}(\omega) &= \frac{1}{2}\left[H_1\left(\frac{\omega}{2}\right)X\left(\frac{\omega}{2}\right) + H_1\left(\frac{\omega - 2\pi}{2}\right)X\left(\frac{\omega - 2\pi}{2}\right)\right] \end{aligned}$$



$$\begin{aligned} V_0(\omega) &= X_{s0}(2\omega) \\ V_1(\omega) &= X_{s1}(2\omega) \end{aligned}$$

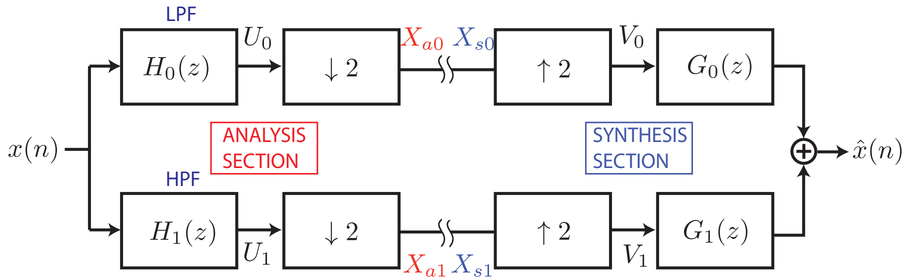
$$\begin{aligned} \hat{X}(\omega) &= V_0(\omega)G_0(\omega) + V_1(\omega)G_1(\omega) \\ &= X_{s0}(2\omega)G_0(\omega) + X_{s1}(2\omega)G_1(\omega) \end{aligned}$$



Suppose we connect the analysis and synthesis sections.

Then,
$$\begin{aligned} X_{s0}(\omega) &= X_{a0}(\omega) \\ X_{s1}(\omega) &= X_{a1}(\omega) \end{aligned}$$

$$\begin{aligned} \hat{X}(\omega) &= X_{s0}(2\omega)G_0(\omega) + X_{s1}(2\omega)G_1(\omega) \\ &= X_{a0}(2\omega)G_0(\omega) + X_{a1}(2\omega)G_1(\omega) \end{aligned}$$



Suppose we connect the analysis and synthesis sections.

$$\begin{aligned}
\hat{X}(\omega) &= X_{a0}(2\omega)G_0(\omega) + X_{a1}(2\omega)G_1(\omega) \\
&= \frac{1}{2}\left[H_0(\omega)X(\omega) + H_0\left(\frac{2\omega-2\pi}{2}\right)X\left(\frac{2\omega-2\pi}{2}\right)\right]G_0(\omega) \\
&+ \frac{1}{2}\left[H_1(\omega)X(\omega) + H_1\left(\frac{2\omega-2\pi}{2}\right)X\left(\frac{2\omega-2\pi}{2}\right)\right]G_1(\omega) \\
&= \frac{1}{2}\left[H_0(\omega)G_0(\omega) + H_1(\omega)G_1(\omega)\right]X(\omega) \\
&+ \frac{1}{2}\left[H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega)\right]X(\omega-\pi)
\end{aligned}$$

$$\begin{aligned}
\hat{X}(\omega) &= \overbrace{\frac{1}{2}\left[H_0(\omega)G_0(\omega) + H_1(\omega)G_1(\omega)\right]X(\omega)}^{\text{desired QMF bank output}} \\
&+ \underbrace{\frac{1}{2}\left[H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega)\right]X(\omega-\pi)}_{\text{effect of aliasing - ELIMINATE!}}
\end{aligned}$$

$$\begin{aligned}
\hat{X}(z) &= \frac{1}{2}\overbrace{\left[H_0(z)G_0(z) + H_1(z)G_1(z)\right]}^{=Q(z)}X(z) \\
&+ \frac{1}{2}\underbrace{\left[H_0(-z)G_0(z) + H_1(-z)G_1(z)\right]}_{=A(z)}X(-z) \\
&= Q(z)X(z) + A(z)X(-z)
\end{aligned}$$

To Eliminate Aliasing

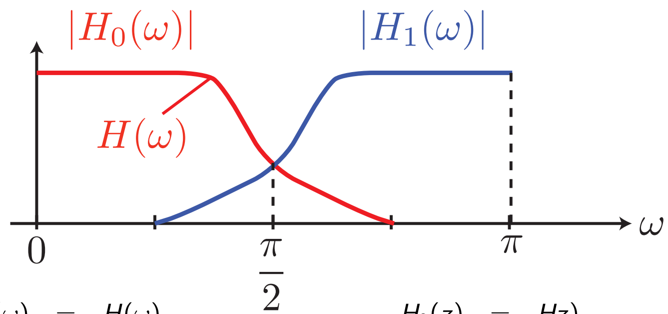
We require $A(z) = 0$ or ...

$$\begin{aligned}
H_0(-z)G_0(z) + H_1(-z)G_1(z) &= 0 \\
\frac{1}{2}\left[H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega)\right] &= 0
\end{aligned}$$

Sufficient condition:
Let $G_0(\omega) = H_1(\omega-\pi)$ and $G_1(\omega) = -H_0(\omega-\pi)$. Therefore,

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}\left[H_0(\omega-\pi)G_0(\omega) + H_1(\omega-\pi)G_1(\omega)\right] \\
&= \frac{1}{2}\left[H_0(\omega-\pi)H_1(\omega-\pi) + H_1(\omega-\pi)(-H_0(\omega-\pi))\right] \\
&= \boxed{0}
\end{aligned}$$

Mirror Image Symmetric Filters



$$\begin{aligned}
H_0(\omega) &= H(\omega) & H_0(z) &= Hz) \\
H_1(\omega) &= H(\omega-\pi) & H_1(z) &= H(-z) \\
G_0(\omega) &= H(\omega) & G_0(z) &= H(z) \\
G_1(\omega) &= -H(\omega-\pi) & G_1(z) &= -H(-z)
\end{aligned}$$

Mirror Image Symmetric Filters

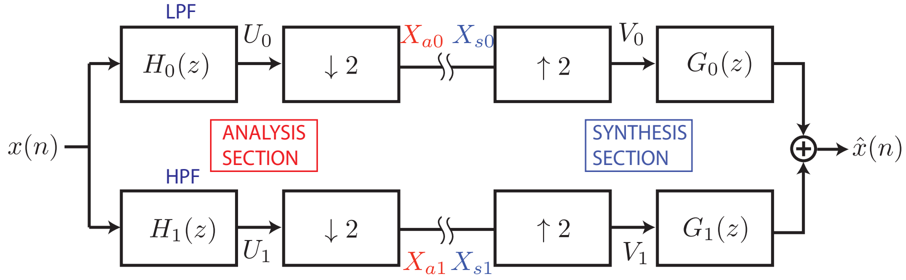
Suppose

$$h(n) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$$

Therefore,

$$\begin{aligned}
H_0(\omega) = H(\omega) &\implies h_0(n) = h(n) \\
H_1(\omega) = H(\omega - \pi) &\implies h_1(n) = e^{j\pi n} h(n) = (-1)^n h(n) \\
G_0(\omega) = H(\omega) &\implies g_0(n) h(n) \\
G_1(\omega) = -H(\omega - \pi) &\implies h_1(n) = -e^{j\pi n} h(n) = (-1)^{n+1} h(n)
\end{aligned}$$

Condition for Perfect Reconstruction



Recall, $\hat{X}(z) = Q(z)X(z) + A(z)X(-z)$. Given that $A(z) = 0$, perfect reconstruction is possible for $\hat{x}(n) = x(n-k)$ or:

$$\begin{aligned}
Q(z) &= \frac{1}{2}[H_0(z)G_0(z) + H_1(z)G_1(z)] = z^{-k} \\
&= \frac{1}{2}[H(z)H(z) + H(-z)(-H(-z))] = z^{-k} \\
\therefore \quad & H^2(z) + H^2(-z) = 2z^{-k}
\end{aligned}$$

Thus,

$$\begin{aligned}
H^2(z) + H^2(-z) &= 2z^{-k} \\
H^2(\omega) - H^2(\omega - \pi) &= 2e^{-j\omega k} \\
|H^2(\omega) - H^2(\omega - \pi)| &= |2e^{-j\omega k}| = 2
\end{aligned}$$

Therefore, $|H^2(\omega) - H^2(\omega - \pi)| = C > 0$ is a necessary condition for perfect reconstruction.

If $H(\omega)$ also has linear phase, this is a sufficient condition for perfect reconstruction.

Polyphase Form of the QMF Bank

Consider polyphase filters for $M = 2$:

$$\begin{aligned}
H_0(z) &= H(z) = P_0(z^2) + z^{-1}P_1(z^2) \\
H_1(z) &= H(-z) = P_0((-z)^2) + (-z)^{-1}P_1((-z)^2) \\
&= P_0(z^2) - z^{-1}P_1(z^2) \\
G_0(z) &= H(z)P_0(z^2) + z^{-1}P_1(z^2) \\
G_1(z) &= -H(-z) = -\left[P_0((-z)^2) + (-z)^{-1}P_1((-z)^2)\right]
\end{aligned}$$

See ‣ Figure 11.11.3 of text. ■

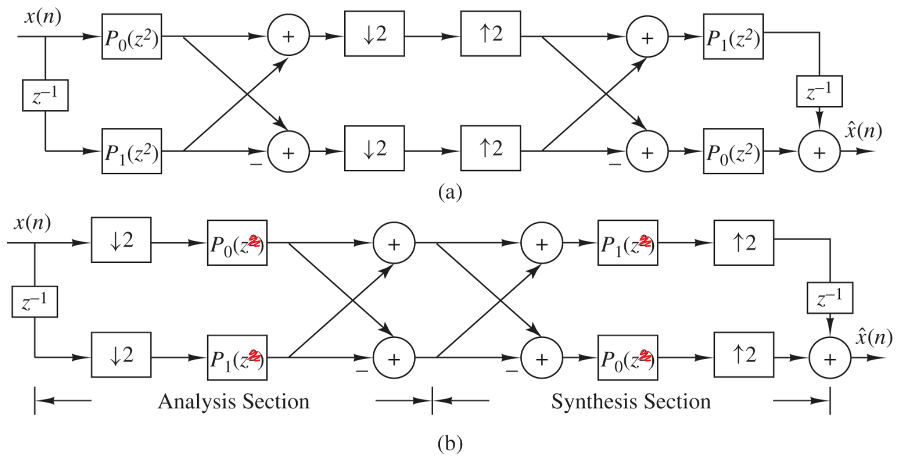


Figure 11.11.3 Polyphase realization of the two-channel QMF bank.

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