Chapter 5: Frequency Domain Analysis of LTI Systems

Discrete-Time Signals and Systems

The Frequency Response Function

- Recall for an LTI system: \( y(n) = h(n) \ast x(n) \).
- Suppose we inject a complex exponential into the LTI system:
  \[
  y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)
  \]
  \[
  x(n) = Ae^{j\omega n}
  \]
- Note: we consider \( x(n) \) to be comprised of a pure frequency of \( \omega \) rad/s

Thus, \( y(n) = H(\omega)x(n) \) when \( x(n) \) is a pure frequency.
Chapter 5: Frequency Domain Analysis of LTI Systems

5.1 Frequency-Domain Characteristics of LTI Systems

The Frequency Response Function

\[ y(n) = H(\omega)x(n) \]

output = scaled input

\[ A \cdot v = \lambda \cdot v \]

LTI System Eigenfunction

- Eigenfunction of a system:
  - an input signal that produces an output that differs from the input by a constant multiplicative factor
  - multiplicative factor is called the eigenvalue

Therefore, a signal of the form \( Ae^{j\omega n} \) is an eigenfunction of an LTI system.

Implications:
- An LTI system can only change the amplitude and phase of a sinusoidal signal.
- An LTI system with inputs comprised of frequencies from set \( \Omega_0 \) cannot produce an output signal with frequencies in the set \( \Omega_0^c \) (i.e., the complement set of \( \Omega_0 \))

Magnitude and Phase of \( H(\omega) \)

\[ H(\omega) = |H(\omega)|e^{j\Theta(\omega)} \]

\[ |H(\omega)| \equiv \text{system gain for freq } \omega \]

\[ \angle H(\omega) = \Theta(\omega) \equiv \text{phase shift for freq } \omega \]
Example:

Determine the magnitude and phase of $H(\omega)$ for the three-point moving average (MA) system

$$y(n) = \frac{1}{3} [x(n + 1) + x(n) + x(n - 1)]$$

By inspection, $h(n) = \frac{1}{3} \delta(n + 1) + \frac{1}{3} \delta(n) + \frac{1}{3} \delta(n - 1)$. Therefore,

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = \sum_{n=-1}^{1} \frac{1}{3} e^{-j\omega n} = \frac{1}{3} (1 + 2 \cos(\omega))$$

Therefore,

$$|H(\omega)| = \frac{1}{3} |1 + 2 \cos(\omega)|$$

$$\Theta(\omega) = \begin{cases} 0 & 0 \leq \omega \leq \frac{2\pi}{3} \\ \pi & \frac{2\pi}{3} \leq \omega < \pi \end{cases}$$

See Figure 5.1.1 of text.

Example:

What is the phase of $H(\omega) = \frac{1}{3}(1 + 2 \cos(\omega))$?

$$|H(\omega)| = \frac{1}{3} |1 + 2 \cos(\omega)|$$

$$\Theta(\omega) = \begin{cases} 0 & 0 \leq \omega \leq \frac{2\pi}{3} \\ \pi & \frac{2\pi}{3} \leq \omega < \pi \end{cases}$$

See Figure 5.1.1 of text.

Frequency Response of LTI Systems

- If $H(z)$ converges on the unit circle, then we can obtain the frequency response by letting $z = e^{j\omega}$:

$$H(\omega) = \lim_{z \rightarrow e^{j\omega}} H(z) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$= \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

for rational system functions.
LTI Systems as Frequency-Selective Filters

- **Filter**: device that discriminates, according to some attribute of the input, what passes through it.

- For LTI systems, given $Y(\omega) = X(\omega)H(\omega)$,
  - $H(\omega)$ acts as a kind of weighting function or spectral shaping function of the different frequency components of the signal.

$\text{LTI system } \iff \text{Filter}$

**Ideal Filters**

- **Classification**:
  - lowpass
  - highpass
  - bandpass
  - bandstop
  - allpass

See Figure 5.4.1 of text.

**Ideal Filters**

- **Common characteristics**:
  - unity (flat) frequency response magnitude in passband and zero frequency response in stopband.
  - linear phase; for constants $C$ and $n_0$

$$H(\omega) = \begin{cases} C e^{-j\omega n_0} & \text{if } \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

- Therefore, $h(n) = C\delta(n - n_0)$ and:
  $$y(n) = x(n) * h(n) = x(n) * C\delta(n - n_0) = Cx(n - n_0)$$
Ideal Filters

- Therefore for ideal linear phase filters:

\[ H(\omega) = \begin{cases} 
C e^{-j\omega_0} & \omega_1 < |\omega| < \omega_2 \\
0 & \text{otherwise}
\end{cases} \]

- signal components in stopband are annihilated
- signal components in passband are shifted (and scaled by passband gain which is unity)

Phase versus Magnitude

What’s more important?
Why Invert?

- There is a fundamental necessity in engineering applications to **undo** the unwanted processing of a signal.
  - reverse intersymbol interference in data symbols in telecommunications applications to improve error rate; called **equalization**
  - correct blurring effects in biomedical imaging applications for more accurate diagnosis; called **restoration/enhancement**
  - increase signal resolution in reflection seismology for improved geologic interpretation; called **deconvolution**
Invertibility of Systems

- Invertible system: there is a one-to-one correspondence between its input and output signals
- The one-to-one nature allows the process of reversing the transformation between input and output; suppose
  \[ y(n) = T[x(n)] \text{ where } T \text{ is one-to-one} \]
  \[ w(n) = T^{-1}[y(n)] = T^{-1}\{T[x(n)]\} = x(n) \]

Invertibility of LTI Systems

- Therefore,
  \[ h(n) * h_I(n) = \delta(n) \]
- For a given \( h(n) \), how do we find \( h_I(n) \)?
- Consider the \( z \)-domain
  \[ H(z)H_I(z) = 1 \]
  \[ H_I(z) = \frac{1}{H(z)} \]

Invertibility of Rational LTI Systems

- Suppose, \( H(z) \) is rational:
  \[ H(z) = \frac{A(z)}{B(z)} \]
  \[ H_I(z) = \frac{B(z)}{A(z)} \]
  - Poles of \( H(z) \) = Zeros of \( H_I(z) \)
  - Zeros of \( H(z) \) = Poles of \( H_I(z) \)
Example

Determine the inverse system of the system with impulse response \( h(n) = \left(\frac{1}{2}\right)^n u(n) \).

- \( H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \), ROC: \(|z| > \frac{1}{2}\), note: direct system is causal + stable.
- Therefore, \( H_1(z) = \frac{1}{H(z)} = 1 - \frac{1}{2}z^{-1} \)
- By inspection, \( h_1(n) = \delta(n) - \frac{1}{2}\delta(n-1) \)
- Is the inverse system stable? causal?

Another Example

Determine the inverse system of the system with impulse response
\( h(n) = \delta(n) - \frac{1}{2}\delta(n - 1) \).

\[
H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} = \sum_{n=-\infty}^{\infty} [\delta(n) - \frac{1}{2}\delta(n-1)]z^{-n}
\]
\[
= 1 - \frac{1}{2}z^{-1}
\]
\[
H_1(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}
\]
Common Transform Pairs

<table>
<thead>
<tr>
<th>Signal, $x(n)$</th>
<th>$z$-Transform, $X(z)$</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(n)$</td>
<td>$1$</td>
<td>All $z$</td>
</tr>
<tr>
<td>$u(n)$</td>
<td>$\frac{1}{1-z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$a^n u(n)$</td>
<td>$\frac{1}{az-1}$</td>
<td>$</td>
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<tr>
<td>$na^n u(n)$</td>
<td>$\frac{1}{az-1}$</td>
<td>$</td>
</tr>
<tr>
<td>$-a^n u(-n-1)$</td>
<td>$\frac{1}{az^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$(\cos(\omega_0 n)) u(n)$</td>
<td>$\frac{1-z^{-1} \cos \omega_0 + z^{-1} \sin \omega_0}{z \sin \omega_0}$</td>
<td>$</td>
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<td>$(a^n \cos(\omega_0 n)) u(n)$</td>
<td>$\frac{1-2az^{-1} \cos \omega_0 + a^2 z^{-2}}{z \sin \omega_0}$</td>
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<td>$</td>
</tr>
</tbody>
</table>

Another Example

There are two possibilities for inverses:

- **Causal + stable inverse:**
  
  \[ h_i(n) = \left( \frac{1}{2} \right)^n u(n) \]

- **Anticausal + unstable inverse:**
  
  \[ h_i(n) = -\left( \frac{1}{2} \right)^n u(-n-1) \]

Homomorphic Deconvolution

- The complex cepstrum of a signal $x(n)$ is given by:
  
  \[ c_x(n) = Z^{-1}\{\ln(Z\{x(n)\})} = Z^{-1}\{\ln(X(z))\} = Z^{-1}\{C_x(z)\} \]

  \[ x(n) \leftrightarrow Z \quad X(z) \]
  
  cepstrum $\equiv c_x(n) \leftrightarrow Z \quad C_x(z) = \ln(X(z))$

- We say, $c_x(n)$ is produced via a **homomorphic transform** of $x(n)$.
Homomorphic Deconvolution

\[ Y(z) = X(z)H(z) \]
\[ C_y(z) = \ln Y(z) \]
\[ = \ln X(z) + \ln H(z) \]
\[ = C_x(z) + C_h(z) \]
\[ Z^{-1}\{C_y(z)\} = Z^{-1}\{C_x(z)\} + Z^{-1}\{C_h(z)\} \]
\[ c_y(n) = c_x(n) + c_h(n) \]

Therefore,

**convolution in time-domain  \iff \ addition in cepstral domain**

Homomorphic Deconvolution

- In many applications, the characteristics of \( c_x(n) \) and \( c_h(n) \) are sufficiently distinct that temporal windows can be used to separate them:

\[
\hat{c}_h(n) = c_y(n)\hat{w}_{lp}(n)
\]
\[
\hat{c}_x(n) = c_y(n)\hat{w}_{hp}(n)
\]

where:

\[
\hat{w}_{lp}(n) = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}
\]
\[
\hat{w}_{hp}(n) = \begin{cases} 0 & |n| \leq N_1 \\ 1 & |n| > N_1 \end{cases}
\]

- Obtaining the inverse homomorphic transforms of \( c_h(n) \) and \( c_x(n) \) give estimates of \( h(n) \) and \( x(n) \), respectively.