Chapter 6: Sampling and Reconstruction of Signals

6.1 Ideal Sampling and Reconstruction of Cts-Time Signals

Analog-to-Digital Conversion

Sampling:
- conversion from cts-time to dst-time by taking "samples" at discrete time instants
- E.g., uniform sampling: \( x(n) = x_a(nT) \) where \( T \) is the sampling period

Sampling Process

- To effectively reconstruct an analog signal from its samples, the sampling frequency \( F_s = \frac{1}{T} \) must be selected to be "large enough".
- Sampling in the time-domain:
  \[
  x(n) = x_a(nT), \quad -\infty < n < \infty
  \]
Sampling Process

- **Time-Domain Sampling: frequency-domain perspective**
  - The sampling frequency $F_s = \frac{1}{T}$ must be selected to be large enough such that the sampling process does not cause any loss of spectral information (i.e., no aliasing).

- Recall CTFT and DTFT for aperiodic $x_a(t)$ and $x(n)$:

$$x_a(t) = \int_{-\infty}^{\infty} X_a(F) e^{j2\pi F t} dF$$

$$x_a(t) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi F t} dt$$

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi F t} dt$$

$$X_a(F) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi F n}$$

We can use the following relationship that comes about from the "reinterpretation stage".

$$t = nT \quad \text{or} \quad \frac{1}{F} = \frac{T}{f} \quad \Rightarrow \quad f = T \cdot F = \frac{F}{F_s} \quad \Rightarrow \quad f = \frac{F}{F_s}$$
Sampling Process

Since,

\[
\frac{1}{F_s} \int_{-F_s/2}^{F_s/2} X\left(\frac{F}{F_s}\right) e^{j2\pi F_n F dF} = \int_{-F_s/2}^{F_s/2} \left[ \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \right] e^{j2\pi F_n F dF}
\]

By inspection,

\[
X\left(\frac{F}{F_s}\right) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s)
\]

or letting \( f = \frac{F}{F_s}\),

\[
X(f) = F_s \sum_{k=-\infty}^{\infty} X_a (f \cdot F_s - kF_s) = F_s \sum_{k=-\infty}^{\infty} X_a ((f - k)F_s)
\]

Sampling Theorem

A bandlimited continuous-time signal, with highest frequency (bandwidth) \( B \) Hz, can be uniquely recovered from its samples provided that the sampling rate \( F_s \geq 2B \) samples per second.

- Perfect reconstruction is possible via the ideal interpolation formula:

\[
x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) \frac{\sin(\pi(t - nT))}{\pi(t - nT)}
\]

Aliasing

- Sampling and reconstruction of nonbandlimited signals results in aliasing.

- The degree of aliasing/quality of the reconstruction depends on the sampling rate in relation to the decay of the analog signal spectrum.

- Example:

\[
x_a(t) = e^{-A|t|} \leftrightarrow X_a(F) = \frac{2A}{A^2 + (2\pi F)^2}, \quad A > 0
\]

See Figure 6.1.8 of text.
Overview

- System setup when discrete-time processing of continuous-time signals is required
- The application often defines how each block is designed

Prefilter

- Ensures that bandwidth is limited to avoid aliasing or reduce subsequent computational requirements
- Rejects additive noise in higher frequency ranges

A/D and D/A

- Ideal sampling and interpolation assumed:
  \[ x(n) = x(t)|_{t=nT} = x_a(nT) \]
  \[ X(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(F-kF_s) \]

  \[ y_a(t) = \sum_{n=-\infty}^{\infty} y(n) \sin(\frac{\pi}{T}(t-nT)) \]
  \[ Y_a(F) = \begin{cases} 
  T Y(F) & |F| \leq \frac{F_s}{2} \\
  0 & \text{otherwise}
  \end{cases} \]

Discrete-Time System Design

- Q: Is there a discrete-time system such that the overall system above is equivalent to a continuous-time LTI system?
- A: Yes if \( x_a(t) \) is bandlimited and \( F_s > 2B \).
Discrete-Time System Design

Consider a desired continuous-time LTI system:

\[ y_a(t) = h_a(t) * x_a(t) = \int_{-\infty}^{\infty} h_a(\tau) x_a(t - \tau) d\tau \]

\[ Y_a(F) = H_a(F) X_a(F) \]

If \( x_a(t) \) is bandlimited and \( F_s > 2B \) (no overlap), then

\[ X(F) = \frac{1}{T} X_a(F) \quad \text{for } |F| \leq \frac{F_s}{2} \]

The desired equivalent cts-time system is:

\[ Y_a(F) = H_a(F) X_a(F) \]

The actual overall response assuming a dst-time filter \( H(F) \) is:

\[ Y_a(F) = \begin{cases} 
T Y(F) & |F| \leq \frac{F_s}{2} \\
0 & \text{otherwise}
\end{cases} \]

\[ Y_a(F) = \begin{cases} 
TH(F) X(F) & |F| \leq \frac{F_s}{2} \\
0 & \text{otherwise}
\end{cases} \]

\[ Y_a(F) = \begin{cases} 
H(F) X_a(F) & |F| \leq \frac{F_s}{2} \\
0 & \text{otherwise}
\end{cases} \]

Therefore,

\[ H_a(F) = \begin{cases} 
H(F) & |F| \leq \frac{F_s}{2} \\
0 & |F| > \frac{F_s}{2}
\end{cases} \]

Naturally,

\[ H(F) = H_a(F) \quad \text{for } |F| \leq \frac{F_s}{2} \]

\[ H(F) = \sum_{k=-\infty}^{\infty} H_a(F - kF_s) \]

\[ h(n) = T \cdot h_a(nT) \]
Chapter 6: Sampling and Reconstruction of Signals  
6.2 Discrete-Time Processing of Continuous-Time Signals

**Discrete-Time System Design**

- Under the conditions discussed, the cascade of a linear time-varying system (A/D converter), an LTI system, and a linear time-varying system (D/A converter) is equivalent to a continuous-time LTI system.

**Example: Ideal bandlimited differentiator**

Choosing $F_s = 2F_c$, we define the ideal discrete-time differentiator as:

$$H(F) = H_a(F) = j2\pi F, \quad |F| \leq \frac{F_s}{2}$$

and

$$H(F) = \sum_{k=-\infty}^{\infty} H_a(F - kF_s)$$

See Figure 6.2.5 of text.

**Bandlimited Signal**

- A continuous-time bandpass signal with bandwidth $B$ and center frequency $F_c$ has its frequency content in two frequency bands defined by $0 < F_L < |F| < F_H$.

$$X_a(F) = \begin{cases} 1 & |F| \leq F_c \\ 0 & |F| > F_c \end{cases}$$

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Note: F_c = \frac{F_L + F_H}{2}, \quad B = F_H - F_L
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**Uniform First-Order Sampling**

\[ x(n) = x_a(nT) \quad \leftrightarrow \quad X(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \]

- \( F_s = 2F_H \) guarantees perfect reconstruction, but wastes bandwidth for bandpass signals

**Integer Band Positioning**

- Consider \( F_H = mB \) where \( B = F_H - F_L \)
- Let \( F_s = 2B \) results in no aliasing
- perfect reconstruction is possible with appropriate interpolation stage

**Integer Band Positioning, \( F_s = 2B, F_H = 4B \)**

**Integer Band Positioning, \( F_s = 2B, F_H = 3B \)**
### Integer Band Positioning

- Perfect reconstruction via:

\[
x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) g_a(t - nT)
\]

\[
g_a(t) = \frac{\sin(\pi Bt)}{\pi Bt} \cos(2\pi F_c t)
\]

for \( F_s = 2B \).

### Arbitrary Band Positioning

- For \( F_s \geq B \), aliasing is due to overlap of “positive” spectral band with “negative” or vice versa.

\[
|X_a(F)|
\]

\[
\frac{1}{T}
\]

\[
F_c
\]

\[
F_L F_c F_H
\]

- The \( k = 1 \) case corresponds to the Nyquist sampling criterion.

- i.e., \( F_s \geq 2F_H \)

\[
2F_H \leq kF_s \implies F_s \geq \frac{2F_H}{k}
\]

\[
\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k - 1}
\]

- If \( F_s \) obeys the above for integer \( k \geq 1 \), aliasing can be avoided.

- The \( k = 1 \) case corresponds to the Nyquist sampling criterion.

- i.e., \( F_s \geq 2F_H \)

- The case \( k > 1 \) corresponds to sampling below Nyquist.

- i.e., \( F_s \leq 2F_L/(k - 1) < 2F_H \) for all \( k > 1 \).

\[
(k - 1)F_s \leq 2F_L \implies F_s \leq \frac{2F_L}{k - 1}
\]
Arbitrary Band Positioning

- The maximum value of $k$ shows the number of bands that we can fit in the range from 0 and $F_H$.

\[(k - 1)F_s \leq 2F_L \implies (k - 1)F_s \leq 2F_H - 2B\]

\[2F_H \leq kF_s \implies \frac{1}{F_s} \leq \frac{k}{2F_H}\]

mult. both sides $\implies k - 1 \leq k - kB\]

\[k \leq \frac{F_H}{B}\]

\[k_{\text{max}} = \left\lfloor \frac{F_H}{B} \right\rfloor\]

Arbitrary Band Positioning: Minimum Sampling Rate

- Recall to avoid aliasing,

\[\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k - 1}\]

- Therefore, the minimum sampling rate to avoid aliasing is given by

\[F_{s,\min} = \frac{2F_H}{k_{\text{max}}}\]

Sampling of Dst-Time Signals

- Therefore to avoid aliasing, the range of acceptable uniform sampling rates is given by

\[\frac{2F_H}{k} \leq F_s \leq \frac{2F_L}{k - 1}\]

where $k \in \mathbb{Z}^+$ and $1 \leq k \leq \left\lfloor \frac{F_H}{B} \right\rfloor$

See Figure 6.4.3 of text.
Sampling of Dst-Time Signals

Consider $x_d(n) = x(nD)$ for all $n$.

- $x(n)$ can be interpreted as the samples of a continuous-time signal $x_a(t)$ with rate $F_s = \frac{1}{T}$:
  \[
  X(F) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(F - kF_s)
  \]

- $x_d(n)$ can be interpreted as the samples of $x_a(t)$ with sampling rate $\frac{F_s}{D}$,
  \[
  X_d(F) = \frac{1}{D \cdot T} \sum_{k=-\infty}^{\infty} X_a(F - k \frac{F_s}{D})
  \]

Therefore,

\[
X_d(F) = \frac{1}{D} \sum_{k=0}^{D-1} X \left( F - k \frac{F_s}{D} \right)
\]

Assuming $X_a(F) = 0, |F| > B$, to avoid aliasing for the dst-time sampling, we need:

\[
\frac{F_s}{D} \geq 2B \quad \text{or} \quad B \leq \frac{F_s}{2D}
\]

\[
f_{\text{max}} \triangleq \frac{B}{F_s} \leq \frac{1}{2D} = \frac{f_s}{2} \quad \text{or} \quad \omega_{\text{max}} = 2\pi f_{\text{max}} \leq \frac{\pi}{D} = \frac{\omega_s}{2}
\]

In cts-time sampling $x(n) = x_a(nT)$,

- the aperiodic spectrum $X_a(F)$ is repeated an infinite number of times to create a periodic spectrum covering the infinite frequency range.

In dst-time sampling $x_d(n) = x(nD)$,

- the periodic spectrum $X(F)$ is repeated $D$ times covering one period of the periodic frequency domain.

Ideal Interpolation

Recall, the general interpolation formula is:

\[
x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \sin(\pi T (t - nT))
\]

For dst-time sampling, it is given by:

\[
x_a(t) = \sum_{m=-\infty}^{\infty} x_d(m) \sin(\pi D T (t - mD T))
\]
Ideal Interpolation

Using

\[ x_d(t) = \sum_{m=-\infty}^{\infty} x_d(m) \frac{\sin\left(\frac{\pi}{D}(t - mD)\right)}{\frac{\pi}{D}(t - mD)} \]

and the fact that \( x(n) = x_a(nT) \)

\[ x(n) = x_a(nT) = \sum_{m=-\infty}^{\infty} x_d(m) \frac{\sin\left(\frac{\pi}{D}(nT - mDT)\right)}{\frac{\pi}{D}(nT - mDT)} \]

\[ = \sum_{m=-\infty}^{\infty} x_d(m) \frac{\sin\left(\frac{\pi}{D}(n - mD)\right)}{\frac{\pi}{D}(n - mD)} \]

\[ = \sum_{m=-\infty}^{\infty} x_d(m) g_{BL}(n - mD) \]

where

\[ g_{BL}(n) = D \frac{\sin\left(\frac{\pi}{D}n\right)}{\pi n} \leftarrow G_{BL}(\omega) = \left\{ \begin{array}{ll} D & |\omega| \leq \frac{\pi}{D} \\ 0 & \frac{\pi}{D} < |\omega| \leq \pi \end{array} \right. \]

See Figure 6.5.1 of text.

Linear Interpolation

See Figure 6.5.2 of text.

\[ x_{lin}(t) = x(m - 1) + \frac{x(m) - x(m - 1)}{DT}(t - (m - 1)DT) \]

for \((m - 1)DT \leq t \leq mDT\)
Linear Interpolation

Therefore,

\[ x_{\text{lin}}(n) = \sum_{m=-\infty}^{\infty} x(m)g_{\text{lin}}(n - mD) \]

\[ g_{\text{lin}}(n) = \begin{cases} 
1 - \frac{|n|}{D} & |n| \leq D \\
0 & |n| > D 
\end{cases} \]

See Figure 6.5.2 of text.

Ideal Interpolation

where

\[ g_{\text{lin}}(n) = \begin{cases} 
1 - \frac{|n|}{D} & |n| \leq D \\
0 & |n| > D 
\end{cases} \leftrightarrow G_{\text{lin}}(\omega) = \frac{1}{D} \left[ \sin\left(\frac{\omega D}{2}\right) \right]^2 \]

See Figure 6.5.3 of text.