The Discrete Fourier Transform

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Discrete-Time Signals and Systems

Reference:

Sections 7.1-7.2 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

Discrete Fourier Transform

- Frequency analysis of discrete-time signals is conveniently performed on a computer.
- Recall:

$$\text{aperiodic in time} \overset{\mathcal{F}}{\longleftrightarrow} \text{continuous in frequency}$$
$$x(n) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

- $X(\omega)$ must, therefore, be stored in samples on a computer.
- What happens when we sample in the frequency domain?

Frequency Domain Sampling

- Recall,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

- Suppose we sample $X(\omega)$.
  - Since $X(\omega)$ is periodic with period 2π, only a finite (say, N) consecutive samples are needed.
  - For convenience, we consider the N equidistant samples in the interval $0 \le \omega \le 2\pi$ with spacing $\delta\omega = \frac{2\pi}{N}$.

Frequency Domain Sampling

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn},$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j2\pi k\frac{n}{N}} \quad \text{Let } n' = n - lN$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n'=0}^{N-1} x(n' + lN) \underbrace{e^{-j2\pi k\frac{n'+lN}{N}}}_{=e^{-j2\pi k\frac{n'}{N}} \underbrace{e^{-j2\pi k\frac{lN}{N}}}_{=1}}$$

Frequency Domain Sampling

For $k = 0, 1, 2, \ldots, N-1$,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n + lN) e^{-j2\pi k\frac{n}{N}}$$

$$= \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n + lN) e^{-j2\pi k\frac{n}{N}}$$

$$= \sum_{n=0}^{N-1} \underbrace{\left[\sum_{l=-\infty}^{\infty} x(n + lN)\right]}_{\text{equivalent signal } x_p(n)} e^{-j2\pi k\frac{n}{N}}$$

Frequency Domain Sampling

For $k = 0, 1, 2, \ldots, N-1$,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n + lN)\right] e^{-j2\pi k\frac{n}{N}}$$

$$= \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi k\frac{n}{N}}$$

- Looks like a DTFS of $x_p(n)$!
- Characteristics of $x_p(n)$:
  - periodic
  - period $= N$
  - can be expanded via a DTFS

Frequency Domain Sampling

DTFS Pair:

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi k\frac{n}{N}}$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi k\frac{n}{N}}$$

Comparing to:

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi k\frac{n}{N}}$$

Frequency Domain Sampling and Reconstruction

Therefore,

$$c_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right) \quad k = 0, 1, \ldots, N-1$$

Since,

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi k \frac{n}{N}} \quad \text{then}$$

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi k \frac{n}{N}} \quad n = 0, 1, \ldots, N-1$$

Frequency Domain Sampling and Reconstruction

Therefore,

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi k \frac{n}{N}} \quad n = 0, 1, \ldots, N-1$$

- Implication: The samples of $X(\omega)$ can be used to reconstruct $x_p(n)$.

and since,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi k \frac{n}{N}} \quad k = 0, 1, \ldots, N-1$$

- Implication: The signal $x_p(n)$ can be used to reconstruct samples of $X(\omega)$.

Frequency Domain Sampling and Reconstruction

$$x(n) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$$

$$x_p(n) \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(\frac{2\pi}{N}k\right)$$

- **FACT:** We can reconstrct $x(n)$ from $X(\omega)$.
- **FACT:** We can reconstrct $x_p(n)$ from samples of $X(\omega)$.

(...and vice versa)

- **Q:** Can we reconstrct $x(n)$ from the samples of $X(\omega)$?
  - x(n) Can we reconstrct $x(n)$ from $x_p(n)$?
- **A:** Maybe.

See Figure 7.1.2 of text.

Frequency Domain Sampling and Reconstruction

- $x(n)$ can be recovered from $x_p(n)$ if there is no overlap when taking the periodic extension.
- If $x(n)$ is finite duration and non-zero in the interval $0 \le n \le L-1$, then

$$x(n) = x_p(n), \quad 0 \le n \le N-1 \quad \text{when } N \ge L$$

- If $N < L$ then, $x(n)$ cannot be recovered from $x_p(n)$.
  - or equivalently $X(\omega)$ cannot be recovered from its samples $X\left(\frac{2\pi}{N}k\right)$ due to time-domain aliasing

Reconstruction, $N \geq L$

- One way to reconstruct $X(\omega)$ from its samples $X\left(\frac{2\pi}{N}k\right)$:
  1. Compute $x_p(n)$ from $X\left(\frac{2\pi}{N}k\right)$:
  $$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{-j2\pi k \frac{n}{N}} \quad n = 0, 1, \ldots, N-1$$
  2. Compute $x(n)$ from $x_p(n)$:
  $$x(n) = \begin{cases} x_p(n) & 0 \leq n \leq N-1 \\ 0 & \text{elsewhere} \end{cases}$$
  3. Compute $X(\omega)$ from $x(n)$:
  $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Reconstruction, $N \geq L$

- Another way to reconstruct $X(\omega)$ from its samples $X\left(\frac{2\pi}{N}k\right)$:

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} x_p(n)e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi k \frac{n}{N}} e^{-j\omega n} \\ &= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(\omega - 2\pi k)\frac{n}{N}}\right] \end{aligned}$$

Reconstruction, $N \geq L$

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) \underbrace{\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(\omega - 2\pi k)\frac{n}{N}}\right]}_{\text{interpolation function}}$$

$$\text{Let } P(\omega) = \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{\sin\left(\omega \frac{N}{2}\right)}{N \sin\left(\frac{\omega}{2}\right)} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$\text{Then } X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) P\left(\omega - \frac{2\pi}{N}k\right) \quad N \geq L$$

See Figure 7.1.3 of text.

Reconstruction

See Figure 7.1.4 of text.

The Discrete Fourier Transform

Summary:

- If $x(n)$ is infinite duration or has length $L>N$, the samples $X\left(\frac{2\pi k}{N}\right), k=0,1,\ldots,N-1$ do not uniquely represent the original sequence $x(n)$.
  - Instead the frequency samples correspond to a periodic sequence $x_p(n)$ of period N where $x_p(n)$ is a time-aliased version of $x(n)$:

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

The Discrete Fourier Transform

Summary:

- If $x(n)$ is infinite duration or has length $L\leq N$, the samples $X\left(\frac{2\pi k}{N}\right), k=0,1,\ldots,N-1$ uniquely represent the original sequence $x(n)$.
  - When $x(n)$ is finite duration of length $L\leq N$, then $x_p(n)$ is a periodic repetition of $x(n)$ that can be recovered from a single period of $x_p(n)$ using:

$$x_p(n) = \begin{cases} x(n) & 0\leq n\leq L-1 \\ 0 & L\leq n\leq N-1 \end{cases}$$

  - Let $X(k) \equiv X\left(\frac{2\pi k}{N}\right)$.

The Discrete Fourier Transform Pair

- DFT and inverse-DFT (IDFT):

$$\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi k \frac{n}{N}}, \quad k=0,1,\ldots,N-1 \\ x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi k \frac{n}{N}}, \quad n=0,1,\ldots,N-1 \end{aligned}$$

DFT Example

Q: Determine the N-point DFT of the following sequence for $N\geq L$:

$$x(n) = \begin{cases} 1 & 0\leq n\leq L-1 \\ 0 & \text{otherwise} \end{cases}$$

A: The DTFT of $x(n)$ is given by:

$$\begin{aligned} X(\omega) &= \sum_{n=0}^{L-1} x(n)e^{-j\omega n} \\ &= \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2} \end{aligned}$$

See Figure 7.1.5 of text.

DFT Example

Thus,

$$X(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

$$X(k) = \frac{\sin(\frac{2\pi k}{N} L/2)}{\sin(\frac{2\pi k}{N}/2)} e^{-j\frac{2\pi k}{N}(L-1)/2} = \frac{\sin(\pi k L/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N}$$

See ▸ Figure 7.1.6 of text. See ▸ Figure 7.1.6b of text.

The DFT as a Linear Transform

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi k \frac{n}{N}}, \quad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi k \frac{n}{N}}, \quad n = 0, 1, \ldots, N-1$$

Want to convert to matrix-vector representation.

The DFT as a Linear Transform

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi k \frac{n}{N}}, \quad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi k \frac{n}{N}}, \quad n = 0, 1, \ldots, N-1$$

Let $W_N = e^{-j2\pi/N}$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad n = 0, 1, \ldots, N-1$$

The DFT as a Linear Transform

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}, \quad n = 0, 1, \ldots, N-1$$

$$\mathbf{x}_N = [x(0)\ x(1)\ \ldots\ x(N-1)]^T$$

$$\mathbf{X}_N = [X(0)\ X(1)\ \ldots\ X(N-1)]^T$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

The DFT as a Linear Transform

- In matrix-vector notation:

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$$
$$\mathbf{x}_N = \mathbf{W}_N^{-1}\mathbf{X}_N = \frac{1}{N}\mathbf{W}_N^{*}\mathbf{X}_N$$

where it can be shown that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^{*}$$

- Complexity: N complex multiplications and $N-1$ complex additions;
  - For an N-point DFT, a total of N^2 complex multiplications and $N(N-1)$ complex additions are required.

Periodicity and Linearity

Notation: $x(n) \overset{N-\text{DFT}}{\longleftrightarrow} X(k)$

- Periodicity:

$$x(n+N) = x(n) \quad \text{for all } n$$
$$X(k+N) = X(k) \quad \text{for all } k$$

- Linearity: If

$$x_1(n) \overset{N-\text{DFT}}{\longleftrightarrow} X_1(k)$$
$$x_2(n) \overset{N-\text{DFT}}{\longleftrightarrow} X_2(k)$$

Then $a_1x_1(n) + a_2x_2(n) \overset{N-\text{DFT}}{\longleftrightarrow} a_1X_1(k) + a_2X_2(k)$

Circular Symmetry and Convolution

- Circular operations: apply the transformation on the periodic repetition of $x(n)$ and then obtain the final result by taking points for $n = 0, 1, \ldots, N-1$
- Circular Symmetry:
  - circular time reversal: $x((-n))_N = x(N-n)$
  - circularly even: $x(N-n) = x(n)$
  - circularly odd: $x(N-n) = -x(n)$
- Circular Convolution:

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N, \quad m = 0, 1, \ldots, N-1$$

Important DFT Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(k)$
Periodicity:	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift:	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift:	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate:	$x^*(n)$	$X^*(N-k)$
Circular convolution:	$x_1(n) \otimes x_2(n)$	$X_1(k)X_2(k)$
Multiplication:	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k) \otimes X_2(k)$
Parseval's theorem:	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

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