Discrete-Time LTI Systems and Analysis

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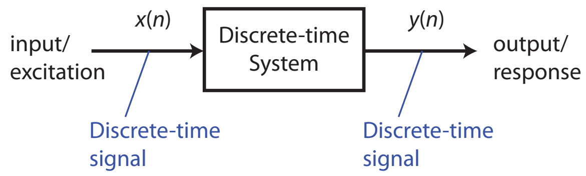
Input-Output Description of Dst-Time Systems



- Input-output description (exact structure of system is unknown or ignored):
$$y(n) = \mathcal{T}[x(n)]$$
- "black box" representation:
$$x(n) \xrightarrow{\mathcal{T}} y(n)$$

System Properties

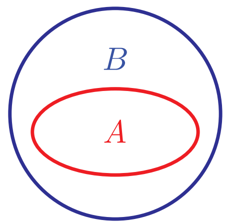
Why is this so important?

- mathematical techniques developed to analyze systems are often contingent upon the general characteristics of the systems being considered
- for a system to possess a given property, the property must hold for every possible input to the system
  - to disprove a property, need a single counter-example
  - to prove a property, need to prove for the general case

Terminology: Implication

If "A" then "B" Shorthand: $A \implies B$

Example 1:
it is snowing $\implies$ it is at or below freezing temperature
Example 2:
$\alpha \geq 5.2 \implies \alpha$ is positive
Note: For both examples above, $B \;\not\Longrightarrow\; A$



Terminology: Equivalence

If "A" then "B" Shorthand: $A \implies B$

$\boxed{\text{and}}$

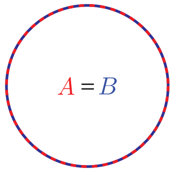
If "B" then "A" Shorthand: $B \implies A$

can be rewritten as

"A" if and only if "B" Shorthand: $A \iff B$

We can also say:

- A is EQUIVALENT to B
- $A = B$

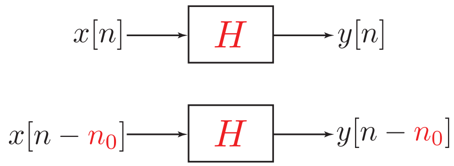


Common Properties

- Time-invariant system: input-output characteristics do not change with time
  - a system is time-invariant iff

$$x(n) \xrightarrow{\mathcal{T}} y(n) \implies x(n-n_0) \xrightarrow{\mathcal{T}} y(n-n_0)$$

for every input $x(n)$ and every time shift n_0.

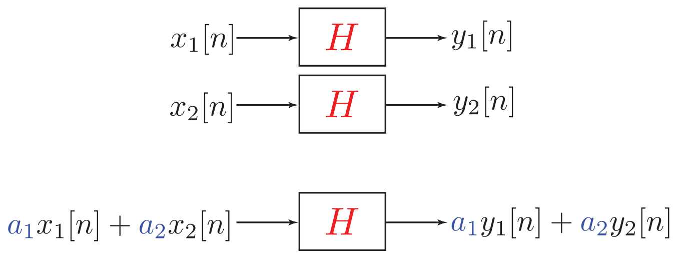


Common Properties

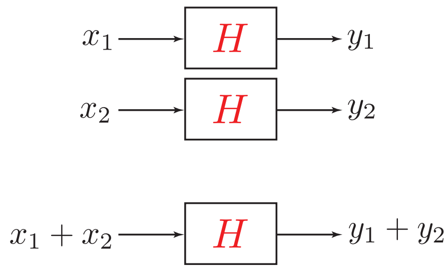
- Linear system: obeys superposition principle
  - a system is linear iff

$$\mathcal{T}[a_1\, x_1(n) + a_2\, x_2(n)] = a_1\, \mathcal{T}[x_1(n)] + a_2\, \mathcal{T}[x_2(n)]$$

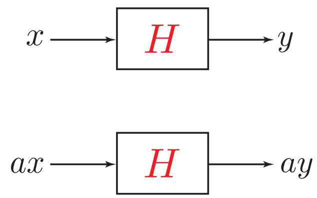
for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.



Additivity:



Homogeneity:



Common Properties

- Causal system: output of system at any time n depends only on present and past inputs
  - a system is causal iff

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots]$$

for all n.

- Bounded Input-Bounded output (BIBO) Stable: every bounded input produces a bounded output
  - a system is BIBO stable iff

$$|x(n)| \leq M_x < \infty \implies |y(n)| \leq M_y < \infty$$

for all n and for all possible bounded inputs.

The Convolution Sum

Recall:

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

The Convolution Sum

Let the response of a linear time-invariant (LTI) system denoted $\mathcal{T}$ to the unit sample input $\delta(n)$ be $h(n)$.

$$\begin{aligned}
\delta(n) &\xrightarrow{\mathcal{T}} h(n) \\
\delta(n-k) &\xrightarrow{\mathcal{T}} h(n-k) \\
\alpha\, \delta(n-k) &\xrightarrow{\mathcal{T}} \alpha\, h(n-k) \\
x(k)\, \delta(n-k) &\xrightarrow{\mathcal{T}} x(k)\, h(n-k) \\
\sum_{k=-\infty}^{\infty} x(k)\delta(n-k) &\xrightarrow{\mathcal{T}} \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
x(n) &\xrightarrow{\mathcal{T}} y(n)
\end{aligned}$$

The Convolution Sum

Therefore,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

for any LTI system.

Causality and Convolution

For a causal system, $y(n)$ only depends on present and past inputs values. Therefore, for a causal system, we have:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \sum_{k=-\infty}^{-1} h(k)x(n-k) + \sum_{k=0}^{\infty} h(k)x(n-k) \\ &= \sum_{k=0}^{\infty} h(k)x(n-k) \end{aligned}$$

where $h(n) = 0$ for $n < 0$ to ensure causality.

Stability and Convolution

It can also be shown that

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \iff \text{LTI system is BIBO stable}$$

Note:

- $\iff$ means that the two statements are equivalent
- BIBO = bounded-input bounded-output

PROOF

For a stable system, $y(n)$ is bounded if $x(n)$ is bounded. What are the implications on $h(n)$?
We have:

$$\begin{aligned} |y(n)| &= \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right| \\ &\leq \sum_{k=-\infty}^{\infty} |h(k)x(n-k)| = \sum_{k=-\infty}^{\infty} |h(k)| \cdot \underbrace{|x(n-k)|}_{|x(n)|\leq M_x<\infty} \\ &\leq \sum_{k=-\infty}^{\infty} |h(k)| M_x = M_x \sum_{k=-\infty}^{\infty} |h(k)| \end{aligned}$$

Therefore, $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$ is a sufficient condition to guarantee:

$$y(n) \leq M_x \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

and we can write:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \implies \text{LTI system is stable}$$

PROOF

To prove the reverse implication (i.e., necessity), assuming $\sum_{n=-\infty}^{\infty} |h(n)| = \infty$ we must find a <u>bounded</u> input $x(n)$ that will always result in an <u>unbounded</u> $y(n)$. Recall,

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ y(0) &= \sum_{k=-\infty}^{\infty} h(k)x(0-k) = \sum_{k=-\infty}^{\infty} h(k)x(-k) \end{aligned}$$

Consider $x(n) = \text{sgn}(h(-n))$; note: $|x(n)| \leq 1$.

$$\begin{aligned} y(0) &= \sum_{k=-\infty}^{\infty} h(k)x(-k) \\ &= \sum_{k=-\infty}^{\infty} h(k)\text{sgn}(h(-(-k))) = \sum_{k=-\infty}^{\infty} h(k)\text{sgn}(h(k)) \\ &= \sum_{n=-\infty}^{\infty} |h(n)| = \infty \end{aligned}$$

PROOF

Therefore,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \infty$$

guarantees that there exists a bounded input that will result in an unbounded output, so it is also a necessary condition and we can write:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \Longleftarrow \quad \text{LTI system is stable}$$

Putting sufficiency and necessity together we obtain:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \Longleftrightarrow \quad \text{LTI system is stable}$$

Note: $\Longleftrightarrow$ means that the two statements are equivalent.

The Direct z-Transform

- Direct z-Transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Notation:

$$X(z) \equiv \mathcal{Z}\{x(n)\}$$

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$$

Region of Convergence

- the region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value
- The z-Transform is, therefore, uniquely characterized by:
  1. expression for $X(z)$
  2. ROC of $X(z)$

z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \lvert z \rvert < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_1(z)$	ROC_2
Linearity:	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$	At least $ROC_1 \cap ROC_2$
Time shifting:	$x(n-k)$	$z^{-k} X(z)$	ROC, except
			$z = 0$ (if $k > 0$)
			and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1} z)$	$\lvert a \rvert r_2 < \lvert z \rvert < \lvert a \rvert r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \lvert z \rvert < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\, x(n)$	$-z \frac{dX(z)}{dz}$	$r_2 < \lvert z \rvert < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z) X_2(z)$	At least $ROC_1 \cap ROC_2$

among others ...

Common Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos(\omega_0 n))u(n)$	$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin(\omega_0 n))u(n)$	$\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos(\omega_0 n)u(n)$	$\frac{1-az^{-1}\cos\omega_0}{1-2az-1\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin(\omega_0 n)u(n)$	$\frac{1-az^{-1}\sin\omega_0}{1-2az-1\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

The System Function

$$\begin{aligned} h(n) &\overset{\mathcal{Z}}{\longleftrightarrow} H(z) \\ \text{time-domain} &\overset{\mathcal{Z}}{\longleftrightarrow} z\text{-domain} \\ \text{impulse response} &\overset{\mathcal{Z}}{\longleftrightarrow} \text{system function} \end{aligned}$$

$$y(n) = x(n) * h(n) \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) = X(z)\cdot H(z)$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)}$$

Discrete-Time Fourier Transform (DTFT)

- DTFT pair:

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega \\ X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \end{aligned}$$

- $X(\omega)$ is the decomposition of $x(n)$ into its frequency components.

Periodicity of the DTFT

Consider

$$\begin{aligned} X(\omega+2\pi) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi)n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\cdot e^{-j2\pi n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\cdot 1 = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega) \end{aligned}$$

Therefore, $X(\omega)$ is periodic with a period of 2π.

Periodicity of the DTFT

- Since $X(\omega) = X(\omega + 2\pi)$, when dealing with discrete frequencies, only a continuous frequency range of length 2π (representing one period) needs to be considered.
  - Minimum frequency for $\omega = 2k\pi$, $k \in \mathbb{Z}$
  - Maximum frequency for $\omega = (2k+1)\pi$, $k \in \mathbb{Z}$
  - Convention is to use $\omega \in [0, 2\pi)$ or $\omega \in (-\pi, \pi]$
- Frequency range of a discrete-time signal is considered to be $\omega \in (-\pi, \pi]$

Periodicity of the DTFT

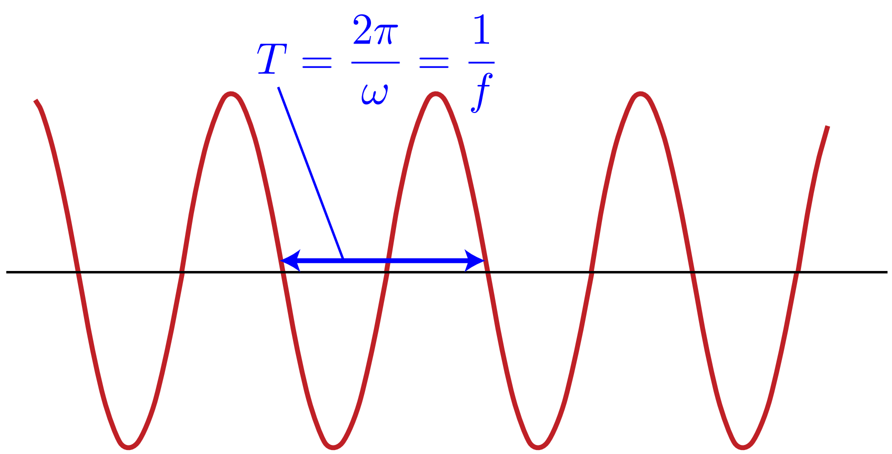
- Continuous-Time Sinusoids: Frequency and Rate of Oscillation:

$$x(t) = A\cos(\omega t + \phi)$$

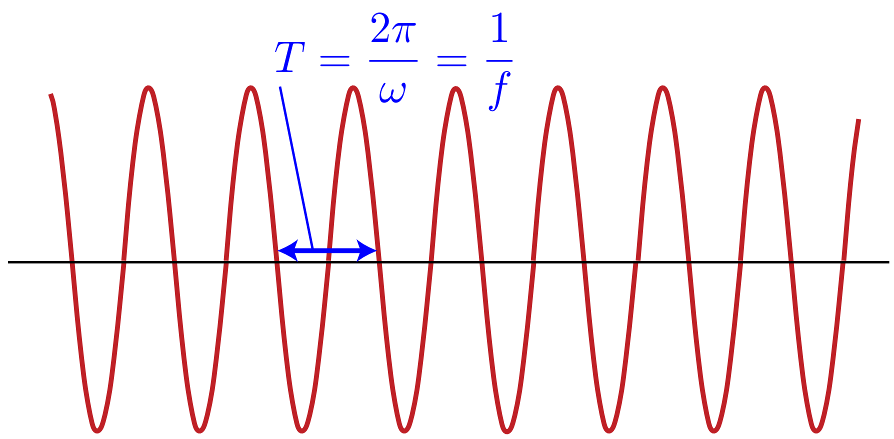
$$T = \frac{2\pi}{\omega} = \frac{1}{f}$$

Rate of oscillation increases as ω increases (or T decreases).

ω smaller



ω larger, rate of oscillation higher

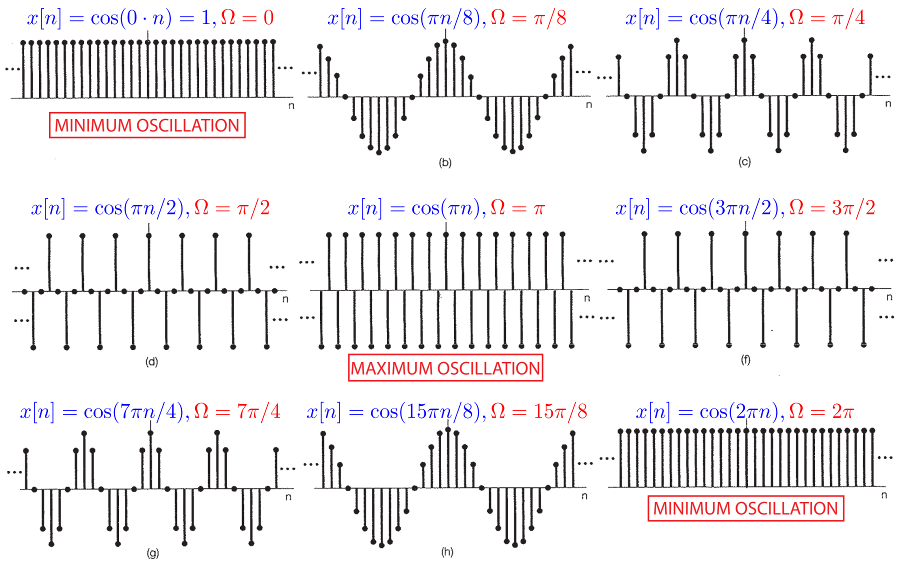


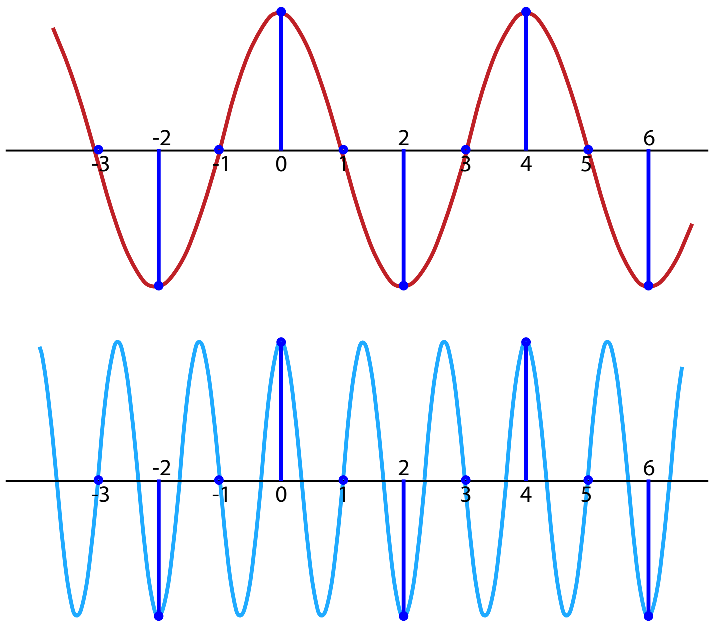
Periodicity of the DTFT

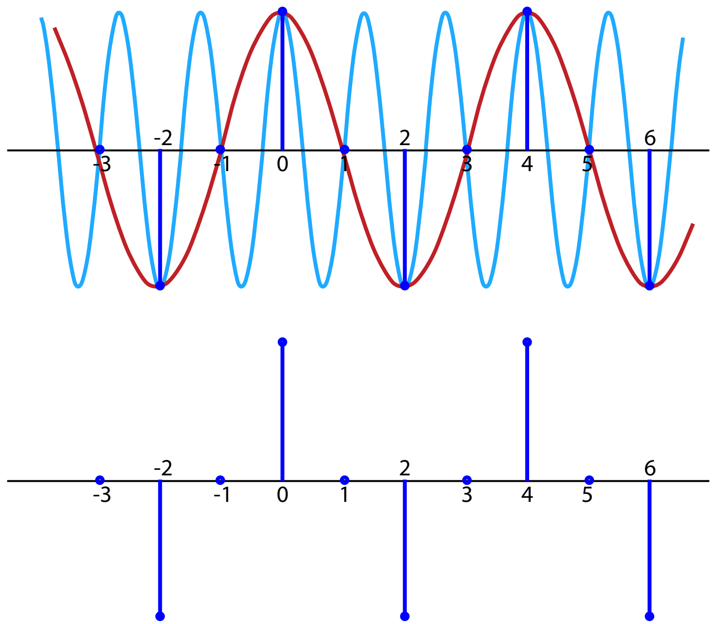
- Discrete-Time Sinusoids: Frequency and Rate of Oscillation:

$$x[n] = A\cos(\Omega n + \phi)$$

Rate of oscillation increases as Ω increases UP TO A POINT then decreases again and then increases again and then decreases again . . . .







DTFT Theorems and Properties

Property	Time Domain	Frequency Domain
Notation:	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_1(\omega)$
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting:	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution:	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation:	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ real]
Wiener-Khintchine:	$r_{xx}(l) = x(l) * x(-l)$	$S_{xx}(\omega) = \lvert X(\omega)\rvert^2$

among others . . .

LTI Filtering

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\
Y(\omega) &= H(\omega)X(\omega)
\end{aligned}
$$

where

$$
\begin{aligned}
x(n) &\stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) \\
h(n) &\stackrel{\mathcal{F}}{\longleftrightarrow} H(\omega) \\
y(n) &\stackrel{\mathcal{F}}{\longleftrightarrow} Y(\omega)
\end{aligned}
$$

$$
\begin{aligned}
H(\omega) &= |H(\omega)|e^{j\Theta(\omega)} \\
|H(\omega)| &\equiv \text{system gain for freq } \omega \\
\angle H(\omega) = \Theta(\omega) &\equiv \text{phase shift for freq } \omega
\end{aligned}
$$

Complex Nature of $X(j\omega)$

Recall, Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad \in \quad \mathbb{C}$$

and Inverse Fourier Transform:

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} X(j\omega)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{\infty} X(j\omega)e^{j\omega t}d\omega
\end{aligned}
$$

<u>Note</u>: If $x(t)$ is real, then the imaginary part of the negative frequency sinusoids (i.e., $e^{j\omega t}$ for $\omega<0$) cancel out the imaginary part of the positive frequency sinusoids (i.e., $e^{j\omega t}$ for $\omega>0$)

Complex Nature of $X(j\omega)$

- Rectangular coordinates: rarely used in signal processing

  $$X(j\omega) = X_R(j\omega) + j\, X_I(j\omega)$$

  where $X_R(j\omega), X_I(j\omega) \in \mathbb{R}$.

- Polar coordinates: more intuitive way to represent frequency content

  $$X(j\omega) = |X(j\omega)|\; e^{j\angle X(j\omega)}$$

  where $|X(j\omega)|, \angle X(j\omega) \in \mathbb{R}$.

Magnitude and Phase of $X(j\omega)$

- $|X(j\omega)|$: determines the relative presence of a sinusoid $e^{j\omega t}$ in $x(t)$
- $\angle X(j\omega)$: determines how the sinusoids line up relative to one another to form $x(t)$

Magnitude and Phase of $X(j\omega)$

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|\; e^{j\angle X(j\omega)}e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|\; e^{j(\omega t+\angle X(j\omega))}d\omega \end{aligned}$$

- Recall, $e^{j(\omega t+\angle X(j\omega))} = \cos(\omega t + \angle X(j\omega)) + j\sin(\omega t + \angle X(j\omega))$.
- The larger $|X(j\omega)|$ is, the more prominent $e^{j\omega t}$ is in forming $x(t)$.
- $\angle X(j\omega)$ determines the relative phases of the sinusoids (i.e. how they line up with respect to one another).

LTI Filtering

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ Y(\omega) &= H(\omega)X(\omega) \end{aligned}$$

where

$$\begin{aligned} x(n) &\stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) \\ h(n) &\stackrel{\mathcal{F}}{\longleftrightarrow} H(\omega) \\ y(n) &\stackrel{\mathcal{F}}{\longleftrightarrow} Y(\omega) \end{aligned}$$

$$\begin{aligned} H(\omega) &= |H(\omega)|e^{j\Theta(\omega)} \\ |Y(\omega)| &= |H(\omega)||X(\omega)| \\ \angle Y(\omega) &= \Theta(\omega) + \angle X(\omega) \end{aligned}$$

LTI Systems as Frequency-Selective Filters

- Filter: device that discriminates, according to some attribute of the input, what passes through it
- For LTI systems, given $Y(\omega) = H(\omega)X(\omega)$
  - $H(\omega)$ acts as a weighting or spectral shaping function of the different frequency components of the signal
  - LTI system is known as a frequency shaping filter

LTI system $\Leftrightarrow$ filter

Causal FIR Filters

Definition: a discrete-time finite impulse response (FIR) filter is one in which the associated impulse response has finite duration.

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \sum_{k=0}^{M-1} h(k)x(n-k) \end{aligned}$$

- lower limit of $k = 0$ is from causality requirement
- upper limit of $0 \leq M-1 < \infty$ is from the finite duration requirement; in this case the support is M consecutive points starting at time 0 and ending at $M-1$

Causal IIR Filters

Definition: a discrete-time infinite impulse response (IIR) filter is one in which the associated impulse response has infinite duration.

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \sum_{k=0}^{\infty} h(k)x(n-k) \end{aligned}$$

- lower limit of $k = 0$ is from causality requirement
- necessary upper limit of ∞ is from the infinite duration requirement

LCCDEs

Linear constant coefficient difference equations (LCCDEs) are an important class of filters that we consider in this course:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

They have a rational system function:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{\text{polynomial in } z}{\text{another polynomial in } z}$$

Depending on the values of N, M, a_k and b_k they can correspond to either FIR or IIR filters.

LCCDEs

Q: Why does an LCCDE have a rational system function?

$$\begin{aligned} y(n) &= -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \\ a_0 y(n) &= -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad a_0 \equiv 1 \\ \sum_{k=0}^{N} a_k y(n-k) &= \sum_{k=0}^{M} b_k x(n-k) \\ \mathcal{Z}\{\sum_{k=0}^{N} a_k y(n-k)\} &= \mathcal{Z}\{\sum_{k=0}^{M} b_k x(n-k)\} \\ \sum_{k=0}^{N} a_k \mathcal{Z}\{y(n-k)\} &= \sum_{k=0}^{M} b_k \mathcal{Z}\{x(n-k)\} \end{aligned}$$

z-Transform Properties

Property	Time Domain	z-Domain	ROC
Notation:	$x(n)$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1(n)$	$X_1(z)$	ROC_1
	$x_2(n)$	$X_1(z)$	ROC_2
Linearity:	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
Time shifting:	$x(n-k)$	$z^{-k}X(z)$	ROC, except $z = 0$ (if $k > 0$) and $z = \infty$ (if $k < 0$)
z-Scaling:	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation:	$x^*(n)$	$X^*(z^*)$	ROC
z-Differentiation:	$n\ x(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution:	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least $\mathrm{ROC}_1 \cap \mathrm{ROC}_2$

among others ...

LCCDEs

$$
\begin{aligned}
\sum_{k=0}^{N} a_k \underbrace{\mathcal{Z}\{y(n-k)\}}_{z^{-k}Y(z)} &= \sum_{k=0}^{M} b_k \underbrace{\mathcal{Z}\{x(n-k)\}}_{z^{-k}X(z)} \\
\sum_{k=0}^{N} a_k z^{-k} Y(z) &= \sum_{k=0}^{M} b_k z^{-k} X(z) \\
Y(z) \sum_{k=0}^{N} a_k z^{-k} &= X(z) \sum_{k=0}^{M} b_k z^{-k} \\
H(z) \equiv \frac{Y(z)}{X(z)} &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad a_0 \equiv 1 \\
H(z) &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 \cdot z^{-0} + \sum_{k=1}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}
\end{aligned}
$$

FIR LCCDEs

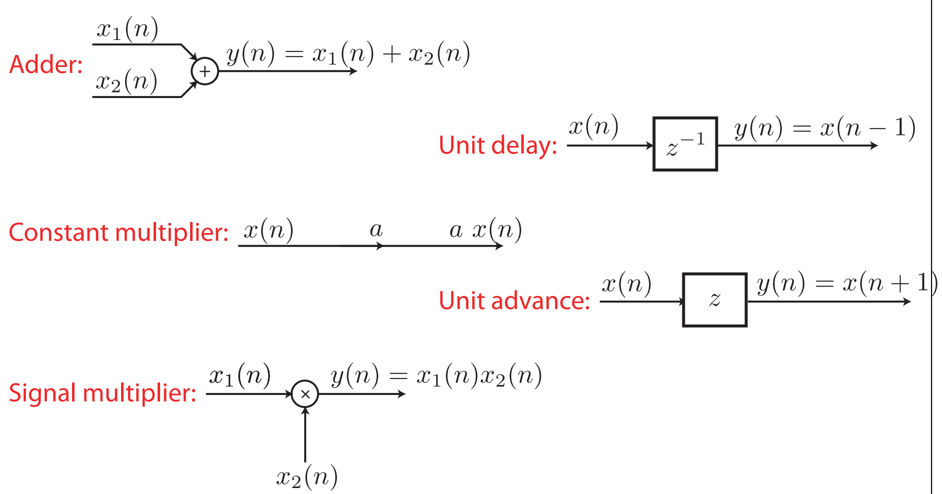
$$
\begin{aligned}
y(n) &= \sum_{k=0}^{M-1} b_k x(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
H(z) &= \sum_{k=0}^{M-1} b_k z^{-k}
\end{aligned}
$$

Please note: upper limit is $M-1$ opposed to M (which is used for the general LCCDE case) to meet common FIR convention of an M-length filter.

By inspection:

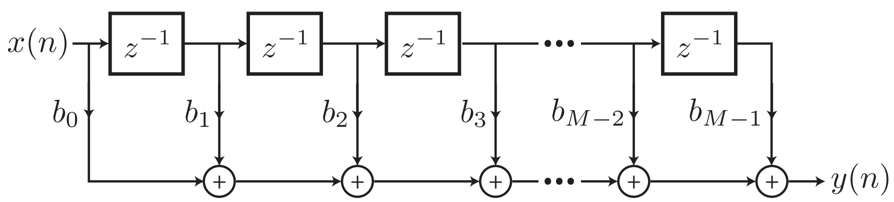
$$
h(n) = \begin{cases} b_n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}
$$

Block Diagram Represenation



FIR Filter Implementation

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

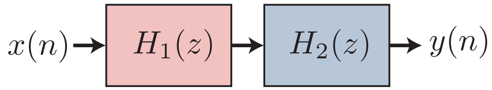


Requires:

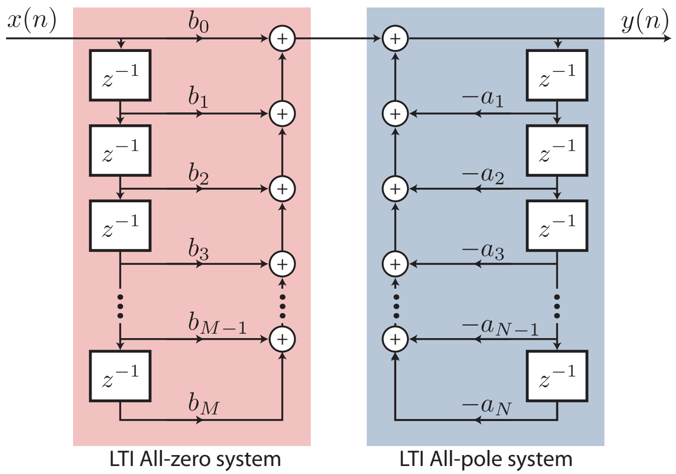
- M multiplications
- $M-1$ additions
- $M-1$ memory elements

IIR LCCDEs

$$\begin{aligned} y(n) &= -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \\ H(z) &= \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = \underbrace{\sum_{k=0}^{M} b_k z^{-k}}_{H_1(z)} \cdot \underbrace{\frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}}_{H_2(z)} \\ &= H_1(z)\cdot H_2(z) \end{aligned}$$

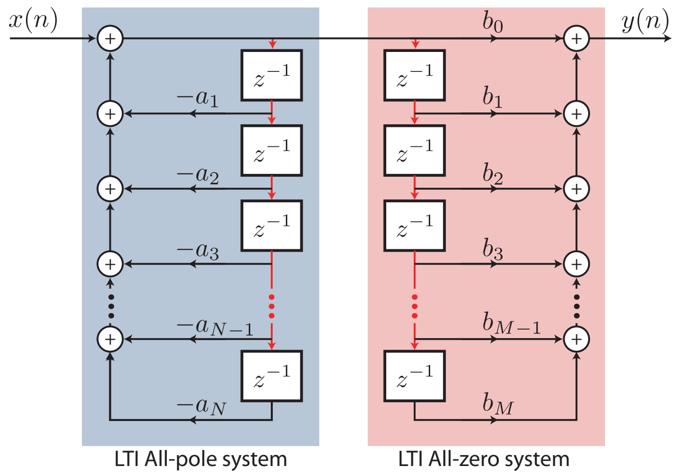


Direct Form I IIR Filter Implementation



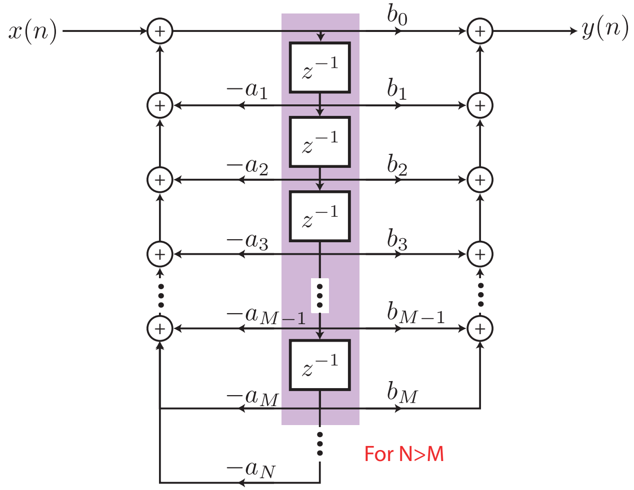
Requires: $M+N+1$ multiplications, $M+N$ additions, $M+N$ memory locations

Direct Form II IIR Filter Implementation



Requires: $M+N+1$ multiplications, $M+N$ additions, $M+N$ memory locations

Direct Form II IIR Filter Implementation



Requires: $M+N+1$ multiplications, $M+N$ additions, $\max(M,N)$ memory locations

Stability of Rational System Function Filters

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Recall, for BIBO stability of a causal system the system poles must be strictly inside the unit circle.

Why?

Stability of Rational System Function Filters

Recall,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \iff \text{LTI system is stable}$$

Stability of Rational System Function Filters

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}|$$

When evaluated for $|z| = 1$ (i.e., on the unit circle),

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Therefore, BIBO stability $\Longrightarrow$ ROC includes unit circle
ROC includes unit circle $\Longrightarrow$ BIBO stability is also true.

Stability of Rational System Function Filters

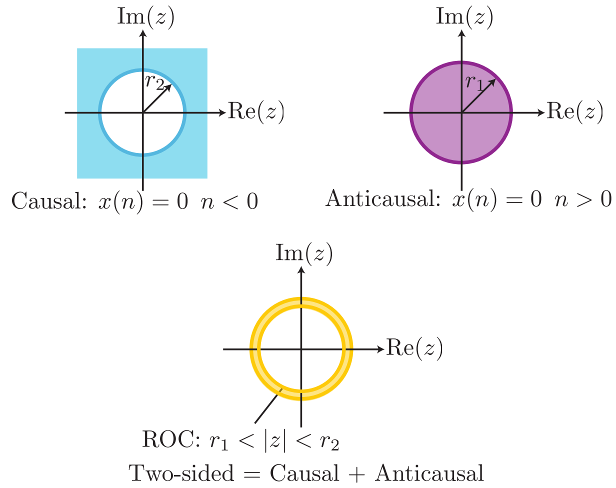
Therefore,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \Longleftrightarrow \quad \text{LTI system is stable} \quad \Longleftrightarrow \quad H(z) \text{ ROC includes unit circle}$$

For a causal rational system function, the ROC includes the unit circle if all the poles are inside the unit circle.

Stability of Rational System Function Filters

. . . because



ARMA, MA and AR Filters

Other commonly used terminology for the filters described include:

- Autoregressive moving average (ARMA) filter:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

- has both poles and zeros
- IIR

ARMA, MA and AR Filters

Other commonly used terminology for the filters described include:

- Moving average (MA) filter:

$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

- has zeros only; no poles; is BIBO stable
- FIR

ARMA, MA and AR Filters

Other commonly used terminology for the filters described include:

- Autoregressive (AR) filter:

$$
\begin{aligned}
y(n) &= -\sum_{k=1}^{N} a_k y(n-k) \\
H(z) &= \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}
\end{aligned}
$$

- has poles only; no zeros
- IIR

■