Effects of Amplitude Quantization

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Quantization

- A digital signal has signal samples represented via finite precision.
- Quantization: process of converting a continuous-amplitude signal to a discrete-amplitude signal
  - used to convert a discrete-time continuous-amplitude signal to a digital signal

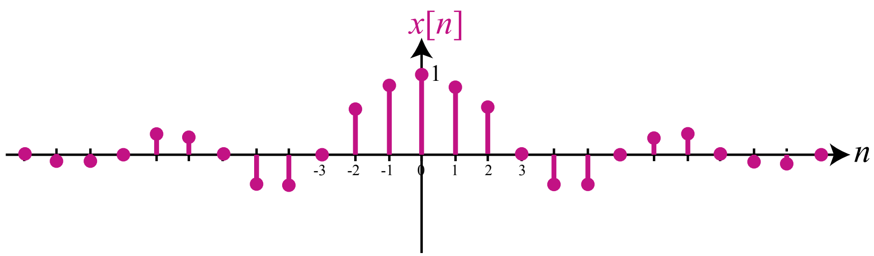
Quantization

Let:

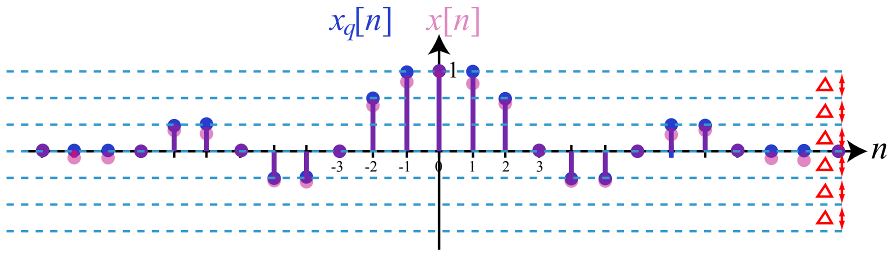
- $x(n)$ = original discrete-time signal
- $x_q(n)$ = quantized digital signal
- $Q[\cdot]$ = quantization operator

$$x_q(n) = Q[x(n)]$$
$$e_q(n) = x_q(n) - x(n)$$

Uniform Quantization Example

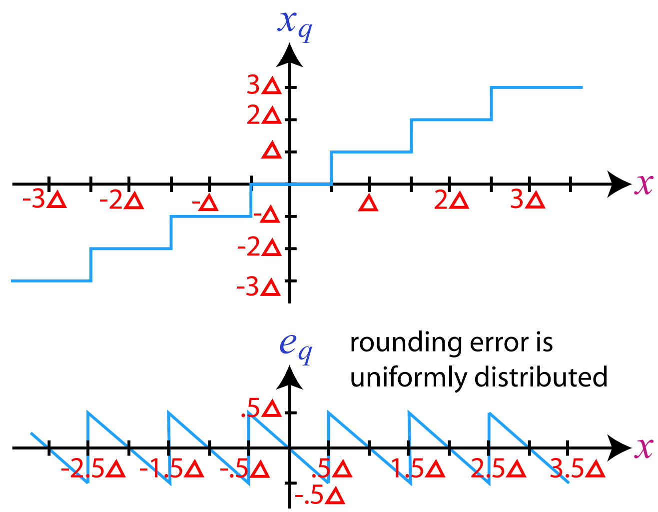


Uniform Quantization Example



... result of rounding to the nearest quantization level.

Uniform Quantization



Quantization Error Statistics: Mean

Probability density function of e_q:

$$p(e) = \begin{cases} \frac{1}{\Delta} & |e| \leq \frac{\Delta}{2} \\ 0 & \text{otherwise} \end{cases}$$

Mean of e_q:

$$\begin{aligned} \mathbf{E}\{e_q\} &\equiv \int_{-\infty}^{\infty} e\ p(e) de = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e\ \frac{1}{\Delta} de \\ &= \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e\ de = \frac{1}{\Delta} \frac{e^2}{2} \Bigg|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{1}{\Delta} \left[\frac{\left(\frac{\Delta}{2}\right)^2 - \left(-\frac{\Delta}{2}\right)^2}{2} \right] = 0 \end{aligned}$$

Quantization Error Statistics: Variance

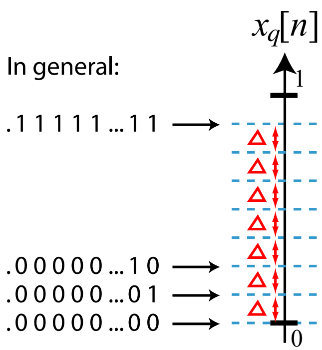
Variance of e_q:

$$\begin{aligned} \mathbf{E}\{(e - \mathbf{E}\{e_q\})^2\} &\equiv \int_{-\infty}^{\infty} (e - \mathbf{E}\{e_q\})^2\ p(e) de \\ &= \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \Big(e - \underbrace{\mathbf{E}\{e_q\}}_{=0} \Big)^2 \frac{1}{\Delta} de = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2\ de \\ &= \frac{1}{\Delta} \frac{e^3}{3} \Bigg|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{1}{\Delta} \frac{\frac{\Delta^3}{8} - \frac{-\Delta^3}{8}}{3} = \frac{\Delta^2}{12} \end{aligned}$$

b-Bit Representation

- Assume the following normalization: $0 \leq x[n] < 1$.
- b-bit unsigned fixed point fractional representation:

$$.\underbrace{x\ x\ x\ x\ x\ x\ x\ x\ x\ \ldots\ x}_{b-\text{bits after the binary point}}$$



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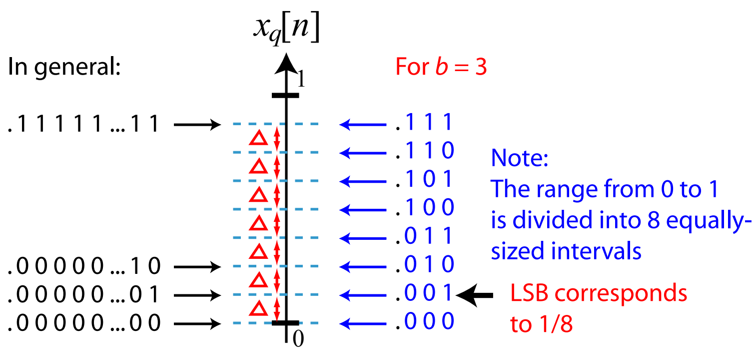

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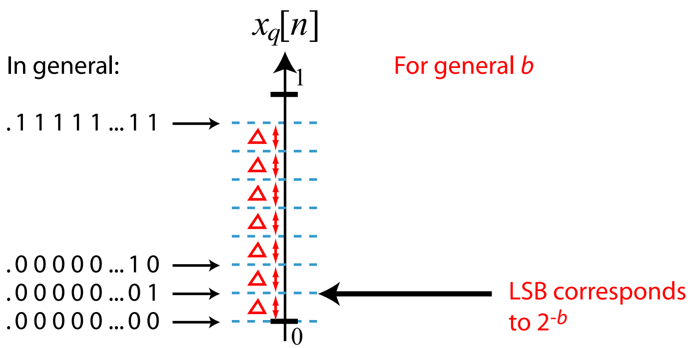


Error Statistics for b-Bit Representation

Mean: For all sample time instants n:

$$\mathbf{E}\{e_q\} = 0$$

Variance: For all sample time instants n:

$$\mathbf{E}\{(e - \mathbf{E}\{e_q\})^2\} = \frac{\Delta^2}{12} = \frac{(2^{-b})^2}{12} = \frac{2^{-2b}}{12}$$

Error Statistics for b-Bit Representation

Autocorrelation:
For $m \neq 0$

$$R_{e_q}(m) = \mathbf{E}\left\{\left(e_q(n) - \underbrace{\mathbf{E}\{e_q(n)\}}_{=0}\right)\left(e_q(n+m) - \underbrace{\mathbf{E}\{e_q(n+m)\}}_{=0}\right)\right\}$$
$$= \mathbf{E}\left\{e_q(n)e_q(n+m)\right\} = \mathbf{E}\left\{e_q(n)\right\}\mathbf{E}\left\{e_q(n+m)\right\} = 0 \cdot 0 = 0$$

For $m = 0$

$$R_{e_q}(0) = \text{variance} = \mathbf{E}\left\{e_q(n)e_q(n+0)\right\} = \mathbf{E}\left\{e_q^2(n)\right\}$$
$$= \frac{\Delta^2}{2} = \frac{2^{-2b}}{12}$$

Therefore,

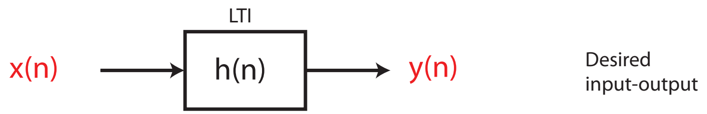
$$R_{e_q}(m) = \begin{cases} \frac{\Delta^2}{2} = \frac{2^{-2b}}{12} & m = 0 \\ 0 & m \neq 0 \end{cases} = \frac{\Delta^2}{2}\delta(m) = \frac{2^{-2b}}{12}\delta(m)$$

Output Quantization

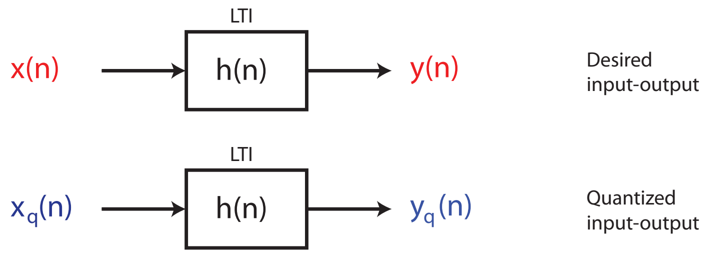
Q: How does an LTI filter affect uniform quantization noise of the input signal?

A: Need to investigate the associated output quantization noise statistics (mean, variance).

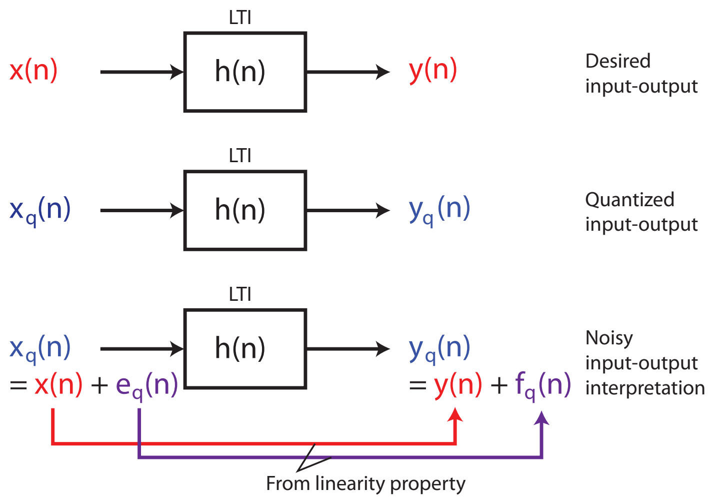
Output Quantization



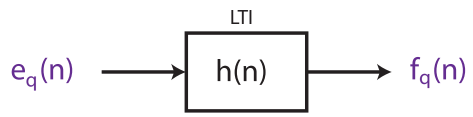
Output Quantization



Output Quantization



Output Quantization Noise Statistics



where $f_q(n) = e_q(n) * h(n)$.

Mean of $f_q(n)$:

$$\mathbf{E}\{f_q(n)\} = \mathbf{E}\{e_q(n) * h(n)\} = \mathbf{E}\left\{\sum_{k=-\infty}^{\infty} e_q(k)h(n-k)\right\}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\mathbf{E}\{e_q(k)\}}_{=0} h(n-k) = 0$$

Output Quantization Noise Statistics

Variance of $f_q(n)$:

$$\mathbf{E}\{(f_q(n) - \underbrace{\mathbf{E}\{f_q(n)\}}_{=0})^2\}$$

$$= \mathbf{E}\{(f_q(n))^2\} = \mathbf{E}\left\{\left(\sum_{k=-\infty}^{\infty} e_q(k)h(n-k)\right)^2\right\}$$

$$= \mathbf{E}\left\{\sum_{k=-\infty}^{\infty} e_q(k)h(n-k) \sum_{l=-\infty}^{\infty} e_q(l)h(n-l)\right\}$$

$$= \mathbf{E}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e_q(k)e_q(l)h(n-k)h(n-l)\right\}$$

Output Quantization Noise Statistics

Variance of $f_q(n)$:

$$\mathbf{E}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} e_q(k)e_q(l)h(n-k)h(n-l)\right\}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \mathbf{E}\{e_q(k)e_q(l)\}h(n-k)h(n-l)$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \underbrace{\mathbf{E}\{e_q(k)e_q(l)\}}_{R_{e_q}(l-k)} h(n-k)h(n-l)$$

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \frac{\Delta^2}{12}\delta(l-k)h(n-k)h(n-l)$$

Output Quantization Noise Statistics

Variance of $f_q(n)$:

$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{\Delta^2}{12}\delta(l-k)h(n-k)h(n-l)$$
$$= \sum_{k=-\infty}^{\infty} \sum_{l \neq k} \frac{\Delta^2}{12}\delta(l-k)h(n-k)h(n-l)$$
$$+ \sum_{k=-\infty}^{\infty} \sum_{l=k} \frac{\Delta^2}{12}\delta(l-k)h(n-k)h(n-l)$$
$$= \sum_{k=-\infty}^{\infty} \frac{\Delta^2}{12} 1 \cdot h(n-k)h(n-k) = \frac{\Delta^2}{12} \sum_{k=-\infty}^{\infty} h^2(n-k)$$

Output Quantization Noise Statistics

Variance of $f_q(n)$:

$$\frac{\Delta^2}{12} \sum_{k=-\infty}^{\infty} h^2(n-k) \qquad \text{let } k' = n-k$$
$$= \frac{\Delta^2}{12} \sum_{k'=-\infty}^{\infty} h^2(k')$$
$$= \frac{\Delta^2}{12} \sum_{n=-\infty}^{\infty} h^2(n) = \frac{\Delta^2}{24\pi} \int_{-\pi}^{\pi} |H(\omega)|^2 d\omega$$

for $h(n)$ real and using Parseval's relation:

$$\sum_{n=-\infty}^{\infty} h^2(n) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega)|^2 d\omega$$

Output Quantization Noise Statistics

Finally for b-bit unsigned fixed point fractional representations ($\Delta = 2^{-b}$):

Mean:

$$\mathbf{E}\{f_q(n)\} = 0$$

Variance:

$$\mathbf{E}\{(f_q(n) - \mathbf{E}\{f_q(n)\})^2\} = \frac{\Delta^2}{12} \sum_{n=-\infty}^{\infty} h^2(n) = \frac{2^{-2b}}{12} \sum_{n=-\infty}^{\infty} h^2(n)$$

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