The Complementary Error Function

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The complementary error function, $\operatorname{erfc}(x)$, is defined, for $x \geq 0$, as

$$\operatorname{erfc}(x) = 2 \int_x^\infty \frac{1}{\sqrt{\pi}} \exp(-u^2)\ \mathrm{d}u.$$

The complementary error function represents the area under the two tails of a zero-mean Gaussian probability density function with variance $\sigma^2 = 1/2$, as illustrated in Fig. 1. The so-called "error function," $\operatorname{erf}(x)$, is defined via

$$\operatorname{erf}(x) = 1 - \operatorname{erfc}(x).$$

From the fact that a probability density function has unit integral, we see that

$$\operatorname{erfc}(0) = 1.$$

The complementary error function $\operatorname{erfc}(x)$ is plotted in Fig. 2, along with an upper and a lower bound (established in Appendix A). The bounds are asymptotically tight, i.e., the difference between the bound and the actual function converges to zero for large x.

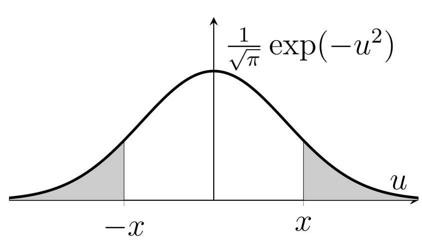

Figure 1: The complementary error function is defined as the area under the two Gaussian pdf tails shown.

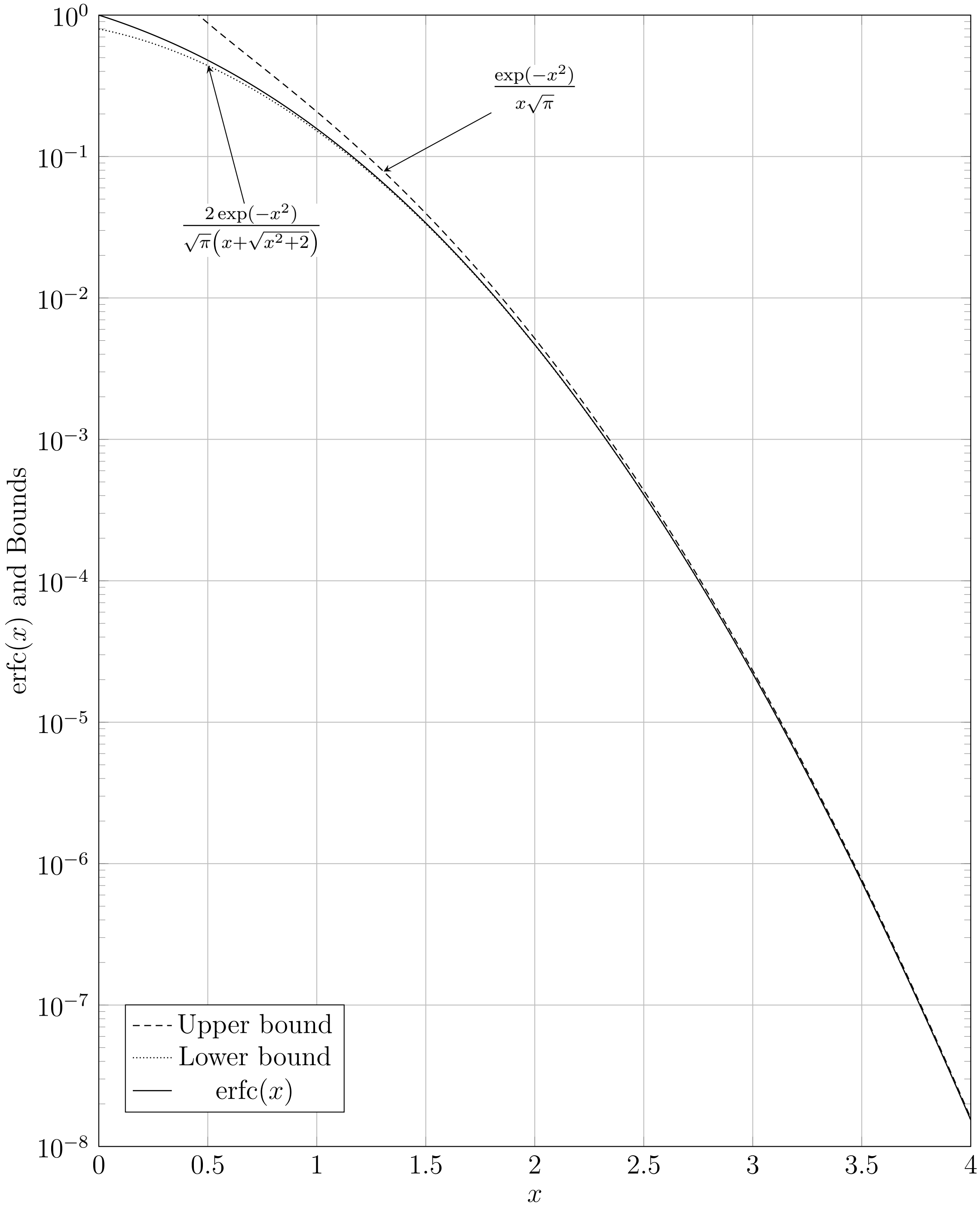

Figure 2: The function erfc(x) plotted together with an upper bound and a lower bound as indicated.

$\mathrm{erfc}(\sqrt{x})$	x	$\mathrm{erfc}(\sqrt{x})$	x	$\mathrm{erfc}(\sqrt{x})$	x	$\mathrm{erfc}(\sqrt{x})$	x
9×10^{-1}	0.007895	9×10^{-5}	7.667845	9×10^{-9}	16.523043	9×10^{-13}	25.525468
8×10^{-1}	0.032092	8×10^{-5}	7.779146	8×10^{-9}	16.637556	8×10^{-13}	25.641072
7×10^{-1}	0.074236	7×10^{-5}	7.905424	7×10^{-9}	16.767404	7×10^{-13}	25.772145
6×10^{-1}	0.137498	6×10^{-5}	8.051325	6×10^{-9}	16.917335	6×10^{-13}	25.923472
5×10^{-1}	0.227468	5×10^{-5}	8.224055	5×10^{-9}	17.094711	5×10^{-13}	26.102474
4×10^{-1}	0.354163	4×10^{-5}	8.435695	4×10^{-9}	17.311866	4×10^{-13}	26.321584
3×10^{-1}	0.537097	3×10^{-5}	8.708911	3×10^{-9}	17.591929	3×10^{-13}	26.604113
2×10^{-1}	0.821187	2×10^{-5}	9.094647	2×10^{-9}	17.986844	2×10^{-13}	27.002402
1×10^{-1}	1.352772	1×10^{-5}	9.755711	1×10^{-9}	18.662447	1×10^{-13}	27.683512
9×10^{-2}	1.437187	9×10^{-6}	9.856363	9×10^{-10}	18.765192	9×10^{-14}	27.787068
8×10^{-2}	1.532451	8×10^{-6}	9.968934	8×10^{-10}	18.880068	8×10^{-14}	27.902841
7×10^{-2}	1.641510	7×10^{-6}	10.096618	7×10^{-10}	19.010323	7×10^{-14}	28.034103
6×10^{-2}	1.768692	6×10^{-6}	10.244100	6×10^{-10}	19.160717	6×10^{-14}	28.185646
5×10^{-2}	1.920729	5×10^{-6}	10.418644	5×10^{-10}	19.338631	5×10^{-14}	28.364901
4×10^{-2}	2.108942	4×10^{-6}	10.632424	4×10^{-10}	19.556431	4×10^{-14}	28.584317
3×10^{-2}	2.354646	3×10^{-6}	10.908279	3×10^{-10}	19.837306	3×10^{-14}	28.867233
2×10^{-2}	2.705947	2×10^{-6}	11.297521	2×10^{-10}	20.233329	2×10^{-14}	29.266055
1×10^{-2}	3.317448	1×10^{-6}	11.964063	1×10^{-10}	20.910728	1×10^{-14}	29.948044
9×10^{-3}	3.411413	9×10^{-7}	12.065497	9×10^{-11}	21.013737	9×10^{-15}	30.051730
8×10^{-3}	3.516737	8×10^{-7}	12.178926	8×10^{-11}	21.128904	8×10^{-15}	30.167647
7×10^{-3}	3.636484	7×10^{-7}	12.307565	7×10^{-11}	21.259485	7×10^{-15}	30.299072
6×10^{-3}	3.775151	6×10^{-7}	12.456124	6×10^{-11}	21.410252	6×10^{-15}	30.450801
5×10^{-3}	3.939719	5×10^{-7}	12.631910	5×10^{-11}	21.588599	5×10^{-15}	30.630273
4×10^{-3}	4.141908	4×10^{-7}	12.847166	4×10^{-11}	21.806920	4×10^{-15}	30.849952
3×10^{-3}	4.403734	3×10^{-7}	13.124855	3×10^{-11}	22.088452	3×10^{-15}	31.133201
2×10^{-3}	4.774768	2×10^{-7}	13.516556	2×10^{-11}	22.485373	2×10^{-15}	31.532483
1×10^{-3}	5.413783	1×10^{-7}	14.186994	1×10^{-11}	23.164238	1×10^{-15}	32.215232
9×10^{-4}	5.511380	9×10^{-8}	14.288989	9×10^{-12}	23.267462	9×10^{-16}	32.319030
8×10^{-4}	5.620616	8×10^{-8}	14.403035	8×10^{-12}	23.382868	8×10^{-16}	32.435073
7×10^{-4}	5.744623	7×10^{-8}	14.532363	7×10^{-12}	23.513717	7×10^{-16}	32.566638
6×10^{-4}	5.887989	6×10^{-8}	14.681703	6×10^{-12}	23.664789	6×10^{-16}	32.718529
5×10^{-4}	6.057833	5×10^{-8}	14.858393	5×10^{-12}	23.843492	5×10^{-16}	32.898190
4×10^{-4}	6.266097	4×10^{-8}	15.074727	4×10^{-12}	24.062243	4×10^{-16}	33.118096
3×10^{-4}	6.535197	3×10^{-8}	15.353760	3×10^{-12}	24.344317	3×10^{-16}	33.401635
2×10^{-4}	6.915542	2×10^{-8}	15.747279	2×10^{-12}	24.741981	2×10^{-16}	33.801318
1×10^{-4}	7.568353	1×10^{-8}	16.420627	1×10^{-12}	25.422064	1×10^{-16}	34.484730

Table 1: Design Table

Table 1 gives a mapping from a desired value of $\operatorname{erfc}(\sqrt{x})$ to the value of x that achieves this value. This table can often be used, in digital communications, to determine the signal-to-noise ratio needed to achieve a target error rate.

The complementary error function is part of the standard math library provided with the `C` programming language (simply `#include <math.h>`) and is also provided by standard math packages such as `Matlab`.

Some digital communications textbooks prefer to define error probabilities in terms of the so-called Q-function, defined, for $x \geq 0$, via

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \mathrm{d}u.$$

This is the area under a *single* tail of a zero-mean Gaussian of a zero-mean Gaussian probability density function with unit variance. The Q-function and the complementary error function are obviously closely related; indeed

$$Q(x) = \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \quad \text{and} \quad \operatorname{erfc}(x) = 2Q\left(x\sqrt{2}\right).$$

A Bounds on the Complementary Error Function

Let X be a Gaussian random variable with probability density function

$$f(x) = \frac{1}{\sqrt{\pi}} \exp(-x^2),$$

and, for $z \geq 0$, let

$$\operatorname{erfc}(z) = 2P[X > z] = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-x^2)\,\mathrm{d}x.$$

Note that $\operatorname{erfc}(0) = 1$. Throughout this appendix, we constrain $z \geq 0$.

For n a non-negative integer, let $M_n(z) = E[X^n \mid X > z]$ denote the conditional nth moment of X, given that $X > z$. Then

$$\begin{aligned}
M_0(z) &= 1 \\
M_1(z) &= \frac{\int_z^\infty \frac{x}{\sqrt{\pi}} \exp(-x^2)\,\mathrm{d}x}{\frac{1}{2}\operatorname{erfc}(z)} = \frac{1}{\sqrt{\pi}\operatorname{erfc}(z)} \int_z^\infty 2x \exp(-x^2)\,\mathrm{d}x \\
&= \frac{\exp(-z^2)}{\sqrt{\pi}\operatorname{erfc}(z)},
\end{aligned}$$

and, for $n \geq 2$, integrating by parts, taking

$$u = x^{n-1} \qquad \mathrm{d}u = (n-1)x^{n-2}\,\mathrm{d}x$$
$$\mathrm{d}v = 2x\exp(-x^2)\,\mathrm{d}x \qquad v = -\exp(-x^2),$$

we get

$$\begin{aligned} M_n(z) &= \frac{\int_z^\infty \frac{x^n}{\sqrt{\pi}}\exp(-x^2)\,\mathrm{d}x}{\frac{1}{2}\operatorname{erfc}(z)} \\ &= \frac{1}{\sqrt{\pi}\operatorname{erfc}(z)}\int_z^\infty 2x\exp(-x^2)x^{n-1}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{\pi}\operatorname{erfc}(z)}\left(z^{n-1}\exp(-z^2) + \frac{n-1}{2}\int_z^\infty 2x^{n-2}\exp(-x^2)\,\mathrm{d}x\right) \\ &= \frac{z^{n-1}\exp(-z^2)}{\sqrt{\pi}\operatorname{erfc}(z)} + \frac{n-1}{2}M_{n-2}(z) \\ &= z^{n-1}M_1(z) + \frac{n-1}{2}M_{n-2}(z). \end{aligned}$$

Thus, for example,

$$\begin{aligned} M_2(z) &= zM_1(z) + \frac{1}{2}, \\ M_3(z) &= z^2M_1(z) + M_1(z) = (z^2+1)M_1(z), \\ M_4(z) &= z^3M_1(z) + \frac{3}{2}M_2(z) = (z^3 + \frac{3}{2}z)M_1(z) + \frac{3}{4}, \\ M_5(z) &= z^4M_1(z) + 2M_3(z) = (z^4 + 2z^2 + 2)M_1(z), \end{aligned}$$

etc.

A simple upper bound on $\operatorname{erfc}(z)$ arises from the observation that $M_1(z) > z$, from which it follows that

$$\operatorname{erfc}(z) < \frac{\exp(-z^2)}{\sqrt{\pi}z}.$$

A lower bound on $\operatorname{erfc}(z)$ arises from the observation that the conditional variance is positive, i.e., $E((X - M_1(z))^2 \mid X > z] = M_2(z) - M_1^2(z) > 0$. We then have

$$zM_1(z) + \frac{1}{2} - M_1(z)^2 > 0$$

Since this parabola in $M_1(z)$ opens downwards, we have that $M_1(z)$ cannot exceed the largest parabolic zero crossing, i.e.,

$$M_1(z) < \frac{z + \sqrt{z^2+2}}{2},$$

from which it follows that

$$\operatorname{erfc}(z) > \frac{2\exp(-z^2)}{\sqrt{\pi}(z + \sqrt{z^2+2})}.$$