The Subadditivity Lemma

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Let $a_1, a_2, a_3, \ldots$ be a sequence of non-negative real numbers with the "subadditive property"

$$a_{i+j} \leq a_i + a_j \text{ for all } i, j \geq 1.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{n}$$

exists and equals $\inf_{n\geq 1}(a_n/n)$.

Proof: Let

$$L = \inf_{n\geq 1} \frac{a_n}{n}.$$

For any $\epsilon > 0$, choose n so that $a_n < n(L + \epsilon)$. (Such an n necessarily exists by definition of infimum.) Let $b = \max_{1\leq i<n} a_i$. If $m \geq n$, let $m = qn + r$ with $0 \leq r < n$. From the subadditivity property,

$$a_{nq+r} = a_{n+n+\cdots+n+r} \leq \underbrace{a_n + a_n + \cdots + a_n}_{q \text{ times}} + a_r \leq qa_n + b.$$

Thus

$$\begin{aligned} \frac{a_m}{m} &\leq \frac{qa_n}{m} + \frac{b}{m} \\ &< \frac{qn(L+\epsilon)}{m} + \frac{b}{m} \\ &\to L + \epsilon \text{ as } m \to \infty \end{aligned}$$

since $qn/m \to 1$ as $m \to \infty$. ∎

For example, let A_n denote the set of sequences—called valid n-sequences—of length n over some finite alphabet. Suppose that, in order for an $(i+j)$-sequence to be valid, it is necessary (but not necessarily sufficient) that its first i components be a valid i-sequence and its last j components be a valid j-sequence. (This situation often arises in constrained coding, for example.) Then $A_{i+j} \subseteq A_i \times A_j$, and hence $|A_{i+j}| \leq |A_i \times A_j| = |A_i| \cdot |A_j|$. Taking logarithms, i.e., defining $a_n = \log_2 |A_n|$, we get

$$a_{i+j} \leq a_i + a_j.$$

The subadditivity lemma guarantees that a_n/n converges to a limit C as $n \to \infty$. Furthermore, $\log_2 |A_n|/n \geq C$ for all n, and, for any $\epsilon > 0$, $\log_2 |A_n|/n < C + \epsilon$ for sufficiently large n. Equivalently,

$$|A_n| \geq 2^{nC} \text{ for all } n$$

and, for all $\epsilon > 0$,

$$|A_n| \leq 2^{n(C+\epsilon)} \text{ for sufficiently large } n.$$

Note: The subadditivity lemma is sometimes called Fekete's Lemma after Michael Fekete [1].

References

[1] M. Fekete, "Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten," *Mathematische Zeitschrift*, vol. 17, pp. 228–249, 1923.