University of Toronto
Faculty of Applied Science and Engineering
Final Exam, May 2010

ECE 466: Computer Networks II
Examiner: J. Liebeherr

• Exam Type: D
  Permitted aids:

  – Two handwritten aid sheets (standard form, only one side of each sheet can be written)
    or one sheet (written on both sides). Content can be any information desired, without
    restriction.

  – Calculator (Type 1).

• All problems have equal weight (10 marks each).
1. **Arrivals and Envelopes.** Consider the following arrival function of a flow:

\[
A(t) = \begin{cases} 
0 , & t \leq 0 \\
t + k , & k \leq t \leq k + 2 \\
2(2t - k - 3) , & k + 2 \leq t \leq k + 3 \\
2t , & k + 3 \leq t \leq k + 5 
\end{cases}
, k = 0, 5, 10, \ldots
\]

(a) (3 marks) Provide a sketch of the arrival function for the interval \([0, 8]\).

(b) (4 marks) Provide a sketch of the empirical envelope \(\mathcal{E}_A\) for the interval \([0, 8]\).

(c) (2 marks) Suppose that \(A(t)\) is the input to a buffered link with capacity \(C = 1\) (data units per time unit). Sketch the departure function \(D(t)\) for the traffic output of the buffered link in the interval \([0, 8]\).

(d) (1 marks) For (c), provide the maximum backlog and maximum delay in the buffered link. (You do not need to perform a computation.)

**Solution:**

(a) See figure.
(b) See figure.
(c) See figure.
(d) See figure. \(B_{\text{max}} = 8\). Since the interval is restricted to \([0, 10]\), the value for \(W_{\text{max}}\) is subject to interpretation. If we assume that at \(t = 10\), all backlogged traffic is serviced instantaneously, we obtain \(W_{\text{max}} = W(5) = 5\). On the other hand, if we assume that - beyond \(t = 10\) - traffic is transmitted at rate \(C = 1\), we get \(W_{\text{max}} = W(10) = 10\).
2. Capacity Requirements.

(a) (5 marks) Consider traffic that is regulated by a leaky bucket with average rate $r = 50 \text{ kbps}$ (kbps = Kilobits per second), and maximum burst size $b = 1000 \text{ bits}$. The regulated traffic is transmitted over a work-conserving buffered link with constant rate $C \text{ kbps}$. Determine the minimum rate $C$ so that the delay at the buffered link does not exceed $10 \text{ msec}$.

(b) (5 marks) Consider traffic that is shaped by a dual leaky bucket with envelope

$$E(t) = \begin{cases} \min\{Pt, b + rt\}, & t > 0 \\ 0, & t \leq 0 \end{cases}, \text{ and } P > r > 0, b > 0.$$ 

The regulated traffic is transmitted on a work-conserving buffered link with constant rate $C \text{ (r < C < P)}$. Determine the minimum rate $C$ so that the backlog at the buffered link does not exceed $B_{\text{max}}$.

Solution:

(a) First get a sense of the solution by drawing the envelope and the service curve. Here, we have a system with an arrival envelope given by a single leaky bucket and a service curve given by a constant-rate link.

In the lecture, we derived an expression for the required capacity that achieves a given maximum delay at a constant-rate link:

$$C_{\text{min}} \geq \sup_{t \geq 0} \left\{ \frac{E(t)}{T + t} \right\}$$

We also derived the solution for a leaky bucket with parameters ($\sigma, \rho$):

$$C_{\text{min}} \geq \sup_{t \geq 0} \left\{ \frac{b + rt}{T + t} \right\} = \max \left\{ \frac{b}{T}, r \right\}$$

Plugging in values, we see:

$$C_{\text{min}} = \max \left\{ \frac{1000 \text{ bits}}{10 \text{ msec}}, 50 \text{ kbps} \right\} = 100 \text{ kbps}$$

(b) From the class notes (Computing Capacity Requirements), we have from

$$B_{\text{max}} \geq \sup_{t \geq 0} \{ E(t) - Ct \}.$$ 

Solving for $C$, this is equivalent to

$$C \geq \sup_{t > 0} \left\{ \frac{E(t) - B_{\text{max}}}{t} \right\}.$$ 

Writing the envelope function $E$ as

$$E(t) = \begin{cases} Pt, & t \leq \frac{b}{P-r} \\ b + rt, & t > \frac{b}{P-r} \end{cases},$$
we see that we should break the supremums above up in two parts. For the backlog we obtain
\[
C \geq \max \left\{ \sup_{t \leq \frac{b}{P-r}} \left\{ \frac{Pt - B_{\text{max}}}{b} \right\}, \sup_{t > \frac{b}{P-r}} \left\{ \frac{b + rt - B_{\text{max}}}{t} \right\} \right\}
\]

Note that the second term is unbounded if \(B_{\text{max}} < b\). Adding the assumption that \(B_{\text{max}} \leq \sigma\) we get
\[
C \geq \max \left\{ P - \frac{B_{\text{max}}(P - r)}{b}, r + \frac{(b - B_{\text{max}})(P - r))}{b} \right\}
= \left( 1 - \frac{B_{\text{max}}}{b} \right) P + \frac{B_{\text{max}}}{b} r .
\]

Thus, the capacity requirements present themselves as a weighted mean of the peak rate \(P\) and the long term rate \(r\). If no memory is available, that is, \(B_{\text{max}} = 0\), the required capacity is the peak rate of the traffic.
3. **Performance Bounds.**

Consider the service element shown in the figure with arrival process $A$ and departure process $D$.

![Diagagram](image)

The server is a delay element that delays each arrival by exactly $T$ time units ($T > 0$). The arrivals $A$ are regulated by an envelope $E$ with

$$E(t) = \begin{cases} \min\{Pt, Rt + B\}, & t > 0 \\ 0, & t \leq 0 \end{cases},$$

with $0 < R < P$, and $B > 0$.

(a) *(3 marks)* Provide a sketch of an arrival function $A$ (which satisfies the envelope $E$) and the corresponding departure function $D$ at the delay element.

(b) *(3 marks)* Derive an expression for the envelope of the departure process $D$ in terms of parameters $P, R, B$ and $T$.

(c) *(4 marks)* Derive a bound for the backlog of this service element, in terms of parameters $P, R, B$ and $T$.

**Solution:**

(a) See Figure. The figure shows an example where $A(t) = E(t)$. Important is that $D(t) = A(t - D)$.

(b) *Since* $D(t) = A(t - T)$ and $A \sim E$, we get that $D E$. So, $E$ as given above is an envelope for $D$.

(c)

$$b^* = E \otimes S(0)$$

$$= \sup_{s \geq 0} \{E(s) - S(s)\}$$

$$= \sup_{s \geq 0} \{\min\{Ps, Rs + B\}I_{s>0} - \delta_T(s)\}$$

$$= \min\{PT, RT + B\}.$$

Note that the supremum is achieved at $s = T$. 

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Consider the network with four service elements. In each network, traffic from a flow with arrival process $A$ passes through the series of network elements.

\[
\begin{array}{cccc}
A \sim E & \rightarrow & S_1 & \rightarrow & S_2 & \rightarrow & S_3 & \rightarrow & S_4 & \rightarrow & D
\end{array}
\]

The network elements are described by the following exact service curves:

$S_1(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ 3(t - 1) & \text{if } t > 1 \end{cases}$, $S_2(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 2t + 2 & \text{if } t > 0 \end{cases}$,

$S_3(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ 2(t - 1) & \text{if } t > 1 \end{cases}$, $S_4(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t + 2 & \text{if } t > 0 \end{cases}$.

Suppose the arrivals $A$ are regulated by a leaky bucket with envelope function $E(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ b + rt & \text{if } t > 0 \end{cases}$, with $r > 0, b > 0$.

(a) (3 marks) Provide sketches of all four service curves and of the envelope function. Give a condition on $b$ and $r$ so that delays and backlog in the network of four servers are finite (i.e., do not become arbitrarily large).

(b) (5 marks) Give an expression for a service curve that describes the service of the entire network. Provide an explanation or derivation of your answer.

(c) (2 marks) For regulated arrivals $A \sim E$, provide a bound on the backlog in the entire network of four service elements in terms of $b$ and $r$. (No derivation is required, but you need to provide the value of the backlog bound. You may assume that the conditions in (a) are satisfied).

Solution:

(a) $b$ can be any number. $r$ must satisfy $r \leq 1$, since the rate of $S_4$ is 1.

(b) The service is expressed by the service curve $S_{net} = S_1 * S_2 * S_3 * S_4$. By associativity we can reorder the service elements to $S_{net} = (S_2 * S_4) * (S_1 * S_3)$.

Since $S_2$ and $S_4$ are both leaky bucket regulators we get (for $t > 0$)

$S_2 * S_4(t) = \min\{S_2(t), S_4(t)\} = \min\{2t + 2, t + 2\} = t + 2$

Also, since $S_1$ and $S_3$ are both latency rate servers we get (for $t > 0$)

$S_1 * S_3(t) = \min\{3, 2\}(t - 1 - 1) = 2(t - 2)$

for $t > 2$ (and otherwise).

So, we have a convolution of a leaky bucket with a latency rate server. This is discussed in Example 2 in Section 1.5.1 in Chapter 1. Let $f = S_2 * S_4$ and $g = S_1 * S_3$. A
quick computation is obtained by exploiting that \( g \) can be written as \( g' \ast \delta_2(t) \) where \( g'(t) = 2t \cdot I_{t>0} \) as follows:

\[
\begin{align*}
f \ast g(t) &= f \ast (g' \ast \delta_2)(t) \\
&= (f \ast g') \ast \delta_2(t) \\
&= \min(2t, t+2) I_{t>0} \ast \delta_2(t) \\
&= \min(2(t-2), t) I_{t>2}.
\end{align*}
\]

(c) The solution can be quickly obtained by graphical inspection. Since \( r < 1 \), the maximum backlog is attained for \( t = 2 \), yielding

\[ b^* = b + 2r \]
5. Fair Bandwidth Allocation.

Consider a link with capacity $C$ that sees arrivals from a set of flows. Each flow $i$ has an arrival rate of $r_i$. Each flow has a weight $\phi_i$.

(a) (5 marks) Given a link with capacity $C = 100$ Mbps, and arrival rates of five flows (given in Mbps):

$$r_1 = 5, r_2 = 15, r_3 = 25, r_4 = 35, r_5 = 40.$$  

All flows have weight $\phi_i = 1$ for $i = 1, \ldots, 5$. Determine the fair share $f$ and the fair bandwidth allocation to each flow.

(b) (5 marks) Given a link with capacity $C = 100$ Mbps, and arrival rates of five flows (given in Mbps):

$$r_1 = 5, r_2 = 15, r_3 = 25, r_4 = 35, r_5 = 40.$$  

The flows have weights:

$$\phi_1 = 1, \phi_2 = 3, \phi_3 = 5, \phi_4 = 5, \phi_5 = 5.$$  

Determine the fair share $f$ and the fair bandwidth allocation to each flow.

Solutions:

(a) Note: The fair share must satisfy

$$\sum_i \min\{r_i, f\} = C.$$  

We get $f = 55/2 = 27.5$  

The allocation is:

$$a_1 = 5, a_2 = 15, a_3 = 25, a_4 = 27.5, a_5 = 27.5.$$  

(b) Note: The fair share must satisfy

$$\sum_i \min\{r_i, \phi_i f\} = C.$$  

Each flow $i$ receives an allocation of

$$a_i = \min\{r_i, \phi_i f\}$$

We get $f = 55/10 \approx 5.5$  

The allocation is:

$$a_1 = 5, a_2 = 15, a_3 = 25, a_4 = 27.5, a_5 = 27.5.$$  

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6. **Bufferless Multiplexer.** Consider a fluid source with a rate distribution $B$ that is uniformly distributed between 0 and the peak rate $R$.

(Hint: The probability density function of $B$ is $f_B(x) = 1/R$, for $0 \leq x \leq R$).

Suppose that $N$ flows share a bufferless multiplexer with capacity $C$. The flows are i.i.d. (independent and identically distributed). Each flow is described by the random variable $B$ defined above.

(a) (4 marks) Show that the mean and variance of the aggregate set of $N$ flows are $RN/2$ and $R^2N/12$, respectively.

(b) (3 marks) Approximate the loss $P_{\text{loss}}$ probability using the central limit theorem. Express your answer in terms of $\Phi(\cdot)$, the distribution function of a $N(0,1)$ random variable.

(c) (4 marks) For $C = 100$ Mbps and $R = 0.1$ Mbps determine the maximum number of flows that can be supported on the link to satisfy a loss probability of $P_{\text{loss}} = 10^{-3}$.

Use the following table to obtain the argument $X$ so that $\Phi(X) \approx 1 - \varepsilon$.

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$1 - \varepsilon$</th>
<th>$X$ with $\Phi(X) \approx 1 - \varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-2}$</td>
<td>0.99</td>
<td>2.33</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>0.999</td>
<td>3.10</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>0.999999</td>
<td>4.88</td>
</tr>
</tbody>
</table>

Solution:

(a)

$$E[B] = \int_0^R \frac{x}{R} \, dx = \frac{x^2}{2R} \bigg|_0^R = \frac{R}{2}$$

$$E[B] = \int_0^R \frac{x}{R} \, dx = \frac{x^2}{2R} \bigg|_0^R = \frac{R}{2}$$

$$E[B^2] = \int_0^R \frac{x^2}{R} \, dx = \frac{x^3}{3R} \bigg|_0^R = \frac{R^2}{3}$$

$$\text{Var}[V] = E[B^2] - (E[B])^2 = \frac{R^2}{3} - \frac{R^2}{4} = \frac{R^2}{12}$$

Let $B_\Sigma = \sum_{j=1}^N B_j$ be the sum of i.i.d. random variables with distribution $B$. We have

$$E[B_\Sigma] = NE[B] = \frac{NR}{2}$$

$$\text{Var}[B_\Sigma] = NV\text{ar}[B] = \frac{NR^2}{12}.$$ 

(b) Thus, the loss probability for $N$ flows in a bufferless multiplexer is given by

$$P_{\text{loss}} = P(\sum_{j=1}^N B_j > C)$$

$$= 1 - \Phi\left(\frac{C - \frac{RN}{2}}{\sqrt{\frac{R^2N}{12}}}\right)$$
(c) From (b) with $\varepsilon = 10^{-3}$, we get

$$C - \frac{RN}{2} = 3.10\sqrt{\frac{R^2N}{12}}$$

which is equivalent to the quadratic equation:

$$N^2 \frac{R^2}{4} - N(RC + \frac{(3.10)^2R^2}{12}) + C^2 = 0$$

With

$$a = \frac{R^2}{4} = 0.0025$$

$$b = (RC + \frac{(3.10)^2R^2}{12}) = 10.008$$

$$c = C^2 = 10000$$

we use the quadratic equation

$$N = \frac{1}{2a} \left( b + \sqrt{b^2 - 4ac} \right) = \frac{1}{2a} \left( b + / - 0.4 \right)$$

$$\approx \min(2, 081.6, 1, 921.6) \approx 1921.6$$

$N$ can be at most 1921.