Delay Bounds for Scheduling Algorithms
First-In-First-Out (FIFO)

\[ \mathcal{N} = \{1, 2, \ldots, N\} \text{ is set of flows} \]
\[ \text{Each flow } j \text{ has a traffic envelope } E_j \ (A_j \sim E_j) \]
\[ \textbf{Goal: } \text{Compute a condition so that the delay of any arrival does not exceed } d. \]
If we set

\[ A(t) := \sum_{k=1}^{N} A_k(t) , \quad E(t) := \sum_{k=1}^{N} E_k(t) , \]

we obtain a buffered link with a single arrival flow.

We have computed this before:

\[ \forall s \geq 0 : E(s - d) \leq Cs \]

Substitute \( s - d \rightarrow s \):

\[
d \geq \sup_{s > 0} \left\{ \frac{E(s) - Cs}{C} \right\}
\]
Static Priority (SP)

- **P** priority classes (1 = lowest, **P** = highest)
- \( \mathcal{N}_p \) is the set of flows with priority \( p \)
- Each flow \( j \) has a traffic envelope \( E_j \) (\( A_j \sim E_j \))
- Ignore fact that packet transmission cannot be preempted.

**Goal:** Compute a condition so that the delay of a **tagged arrival** at time from flow \( j \in \mathcal{N}_p \) at time \( t^- \) has a delay less than \( d_p \).
Static Priority

- $t - x_p$: last time before $t$ that the SP scheduler does not have any backlog from priority-$p$ or higher.
  - From $t - x_p$, until tagged arrival leaves, scheduler only transmits traffic from priority $p$ or higher.
  - $A_j(t - x_p) = D_j(t - x_p)$
Notation: \( A_j(s, t) = A_j(t) - A_j(s), \) \( D_j(s, t) = D - j(t) - D_j(s) \)

Delay of tagged arrival:

\[
W_j(t) = \inf \{ y > 0 \mid D_j(t + y) \geq A_j(t) \} \\
= \inf \{ y > 0 \mid D_j(t - x_p, t + y) \geq A_j(t - x_p, t) \}
\]

\[
D_j(t - x_p, t + y) \leq \left[ \{ \text{Total traffic transmitted in } [t - x_p, t + y) \} \\
- \{ \text{Arrivals in } [t - x_p, t + y) \text{ with higher precedence} \} \right]^+
\]

\[
= \left[ C(y + x_p) - \sum_{k \in \mathcal{N}_p, k \neq j} A_k(t - x_p, t) - \sum_{q=p+1}^{P} \sum_{k \in \mathcal{N}_q} A_k(t - x_p, t + y) \right]^+\]
SP Analysis (2)

- Replace $D_j$ in expression for $W_j$ by its bound:

$$W_j(t) \leq \inf \left\{ y > 0 \mid C(y + x_p) - \sum_{k \in \mathcal{N}_p, k \neq j} A_k(t - x_p, t) - \frac{P}{q=p+1} \sum_{k \in \mathcal{N}_q} A_k(t - x_p, t + y) \geq A_j(t - x_p, t) \right\}.$$ 

- If $d_p$ is a delay bound, then $W_j(t) \leq d_p$. So:

$$C(d_p + x_p) - \sum_{k \in \mathcal{N}_p, k \neq j} A_k(t - x_p, t) - \sum_{q=p+1}^P \sum_{k \in \mathcal{N}_q} A_k(t - x_p, t + d_p) \geq A_j(t - x_p, t).$$
Re-arrange terms and allow \( x_j \) to be any value \( s \geq 0 \):

\[
Cd_p \geq \sup_{s \geq 0} \left\{ \sum_{k \in \mathcal{N}_p} A_k(t-s, t) + \sum_{q=p+1}^{P} \sum_{k \in \mathcal{N}_q} A_k(t-s, t+d_p) - Cs \right\}.
\]

Relax expression

- Set \( s = x_p \)
- Use that \( A_k(s, t) \leq E_k(t - s) \)

Simplify: \( E_q(t) = \sum_{j \in \mathcal{N}_q} E_j(t) \)

\[
d_p \geq \sup_{s \geq 0} \frac{1}{C} \left\{ E_p(s) + \sum_{q=p+1}^{P} E_q(s + d_p) - Cs \right\}
\]

If condition is satisfied, tagged arrival has delay no more than \( d_p \).
Each flow $j$ has a delay index $d_j$.

Each flow $j$ has a traffic envelope $E_j$ ($A_j \sim E_j$)

**Goal:** Compute a condition so that the delay for a tagged arrival at time from flow $j \in \mathcal{N}_p$ at time $t$ does not exceed $d_j$ (→ Tagged arrival meets its deadline!)
Q: Which traffic has higher precedence than tagged arrival?
A: All traffic with a deadline before $t + d_j$!

- for flow $j$: All arrivals before $t$
- for any flow $k$: All arrivals before $t + d_j - d_k$

If the tagged arrival has not departed by time $t + y$, traffic from flow $k$ with deadline $\leq t + d_j$ or earlier, are the arrivals until time $t + \min\{y, d_j - d_k\}$.
Define $t - x_j$: last time before $t$ that the EDF scheduler does not have any backlog with deadline $t + d_j$ or earlier

- From $t - x_j$, until tagged arrival leaves, scheduler only transmits traffic with deadline $\leq t + d_j$
- $A_j(t - x_j) = D_j(t - x_j)$
EDF Analysis (1)

- Delay of tagged arrival:

\[
W_j(t) = \inf \{ y > 0 \mid D_j(t + y) \geq A_j(t) \} = \inf \{ y > 0 \mid D_j(t - x_j, t + y) \geq A_j(t - x_j, t) \}.
\]

- If tagged arrival has not left by time \( t + y \):

\[
D_j(t - x_j, t + y) \leq \left[ \left\{ \text{Total traffic transmitted in } [t - x_j, t + y) \right\} - \left\{ \text{Arrivals from flow } j \text{ in } [t - x_j, t + \min\{y, d_j - d_k\}) \right\} \right]^+ \\
= \left[ C(y + x_j) - \sum_{k \in \mathcal{N}, k \neq j} A_k(t - x_j, t + \min\{y, d_j - d_k\}) \right]^+.
\]
EDF Analysis (2)

- Replace $D_j$ in expression for $W_j$ by its bound:

$$W_j(t) \leq \inf \{ y > 0 \mid C(y + x_j)$$

$$- \sum_{k \in N, k \neq j} A_k(t - x_j, t + \min\{y, d_j - d_k\}) \geq A_j(t - x_j, t) \}$$

- The delay index $d_j$ is a delay bound, if $W_j(t) \leq d_j$. So:

$$C(d_j + x_j) - \sum_{k \in N, k \neq j} A_k(t - x_j, t + \min\{d_j, d_j - d_k\}) \geq A_j(t - x_j, t) .$$
Re-arrange terms and allow $x_j$ to be any value $s \geq 0$:

$$\sup_{s \geq 0} \left\{ \sum_{k \in \mathcal{N}} A_k(t - s, t + d_j - d_k) - C(d_j + s) \right\} \leq 0 .$$

Use that $A_k(s, t) \leq E_k(t - s)$

$$\sup_{s \geq 0} \left\{ \sum_{k \in \mathcal{N}} E_k(s + d_j - d_k) - C(s + d_j) \right\} \leq 0 .$$

Substitute $s + d_j \rightarrow s$

$$\sup_{s \geq 0} \left\{ \sum_{k \in \mathcal{N}} E_k(s - d_k) - Cs \right\} \leq 0$$

If condition is satisfied, tagged arrival departs before its deadline $t + d_j$. 
The importance of a delay bound analysis for link schedulers is that it can be used to devise tests that determine whether a scheduler can satisfy given delay requirements for a set of flows at a link scheduler with capacity $C$.

Conditions that verify whether a set of flows is schedulable are called *schedulability conditions*. The derived inequalities are schedulability conditions for FIFO, SP, and EDF. Conditions are ‘tight’, i.e., violation of the conditions may result in violation of delay bound.