1. Derive the deconvolution $f \circ g$ for functions $f$ and $g$ given by:

$$f(t) = \begin{cases} 
0, & t \leq 0 \\
t + 3, & t > 0 
\end{cases}, \quad g(t) = \begin{cases} 
0, & t \leq 0 \\
2t + 1, & t > 0 
\end{cases}.$$

**Solution:**

The solution is illustrated in the figure:

Writing out the deconvolution, we have

$$f \circ g(t) = \sup_{s \geq 0} \{(t + s + 3)I_{t+s>0} - (2s + 1)I_{s>0}\}.$$

We first consider the case $t > 0$ and then $t \leq 0$. For positive values of $t$, we can see that the first indicator function is active in the entire range $s \geq 0$, and that the second indicator function is active for $s > 0$. So, for $s = 0$ we have only $t + 3$ inside the supremum. For $s > 0$, we get $t + s + 3 - (2s + 1)$. So we have for $t > 0$ that

$$f \circ g(t) = \max \left\{ t + 3, \sup_{s>0} \{t - s + 2\} \right\}$$

$$= \max \{t + 3, t + 2\}$$

$$= t + 3.$$
Now let us consider the case \( t \leq 0 \). Here, the indicator functions change at time \( s = 0, \ 0 < s \leq -t, \) and \( s > -t \). Evaluating the expressions in the respective ranges we obtain for \( t \leq 0 \) that

\[
f \otimes g(t) = \max \left\{ 0, \sup_{0 < s \leq -t} \{-2s - 1\}, \sup_{s > -t} \{t - s + 2\} \right\}
\]

\[
= \max \{0, -1, 2t + 2\}
\]

\[
= \max \{2t + 2, 0\}.
\]

The result of the derivation is therefore

\[
f \otimes g(t) = \begin{cases} 
0 , & t \leq -1 \\
2t + 2 , & -1 < t \leq 0 \\
t + 3 , & t > 0
\end{cases}
\]

The function is illustrated in Figure 2. Since \( f \otimes g \) is positive in the range \([-1, 0]\), the deconvolution of these functions is not a one-sided process.

2. Consider the network with two constant rate links (with rates \( C_1 \) and \( C_2 \)) and a delay element (with delay \( W \)) as shown in Figure 1.

![Figure 1](image)

Suppose that the arrivals from \( A \) are regulated by a dual leaky bucket with peak rate \( P = 1 \) Mbps, average rate \( \rho = 0.5 \) Mbps and burst size \( \sigma = 15,000 \) bits. Also, suppose that \( W = 10 \) ms.

Determine the minimum rates for \( C_1 \) and \( C_2 \) so that the backlog in the network does not exceed 1000 bits.

**Solution:** NOTE: Review Problem Set 3, Problem 1.

We need to compute or recall the maximum backlog in a constant rate server with a dual leaky bucket.

The maximum backlog is maximum vertical difference between the envelope of the dual leaky bucket \( E(t) = \min( Pt, \sigma + pt ) \) and the service curve \( S_{net}(t) = \min(C_1, C_2)(t - W)^+ \). This is illustrated in the figure.
Figure 2: \[ x^* = \frac{\sigma}{P - \rho} \]
If we look at the vertical difference between the two graphs in Figure 2, we note that the behavior is different for $t < W$, $W \leq t < x^*$ and $t \geq x^*$. (The solution to the problem will be different for $W < x^*$ and $W > x^*$. Since we have $W = 10$ ms and $x^* = \frac{\sigma}{P - \rho} = 30$ ms we are dealing with the case $W < x^*$.)

Applying the expression for the maximum backlog we derive:

$$b_{\text{max}} = E \bigcirc S(0)$$
$$= \sup_{s\in \mathbb{R}}(E(s) - S(s))$$
$$= \sup_{s\in \mathbb{R}}(\min(Ps, \sigma + ps) - C(s - W)^+)$$
$$= \max\left(\sup_{s<W} (P \cdot s), \sup_{W \leq s < x^*} (P \cdot s - C(s - W)), \sup_{s \geq x^*} (\sigma + ps - C(s - W))\right)$$

$$= \max\left(P \cdot W, \left\{ \begin{array}{ll}
\sigma \frac{P-C}{PW} + CW & \text{if } P \geq C \\
\frac{P-C}{P-P-\rho} & \text{if } P < C \end{array} \right. \right)$$

From this derivation we see that

$$b_{\text{max}} \geq P \cdot W = 10,000 \text{ bit}.$$ 

This means that the backlog requirements of $b_{\text{max}} \leq 1,000$ bit cannot be satisfied through an appropriate selection of $C_1$ and $C_2$.

3. Consider the traffic of an MPEG video stream that is regulated with a leaky bucket with average rate $\rho = 0.15 \text{ Mbps}$ (Mbps = Megabits per second), and maximum burst size size $\sigma = 15,000 \text{ bits}$.

(a) The output of the regulator feeds into a constant rate link with capacity $C = 300 \text{ kbps}$. Determine the maximum backlog and the maximum delay at the link.

(b) Suppose that the regulator is a dual leaky bucket with peak rate $P = 1 \text{ Mbps}$ (in addition to $\rho = 0.15 \text{ Mbps}$ and $\sigma = 15,000 \text{ bits}$). Determine the maximum backlog and the maximum delay at the link.

(c) Continue with the assumptions in (b). Instead of a single link, now assume that the traffic goes through a sequence of three links with rates $C_1 = 400 \text{ kbps}$, $C_2 = 200 \text{ kbps}$, and $C_3 = 300 \text{ kbps}$, for the first, second and third link, respectively. Determine the maximum backlog and the maximum delay in the network.
Solution:
You get an idea of the solution if you first draw the envelope and
the service curve. We know that the max. backlog is the maximum
vertical distance between the two curves, and the max. delay is the
maximum horizontal distance between the curve.

(a) We have an envelope for the arrivals of \( R(t) = \sigma + \rho t \), and a
service curve of \( S(t) = Ct + \). 
From the lecture we know that
\[
b_{\text{max}} = E \cap S(0) = \sup_{t \geq 0} (\sigma + \rho t - Ct)
\]
For the given envelope, we see that the expression is decreasing
in \( t \) (since \( \rho < C \)). In other words, the largest value (sup) is
attained when \( t \) is selected as small as possible \( (t = 0) \). We get:
\[
b_{\text{max}} = \sigma = 15,000 \text{ bits}
\]
To get the max. delay, look up your lecture notes and find
\[
d_{\text{max}} = \sup_{t \geq 0} \left( \frac{E(t) - S(t)}{C} \right)
\]
Plugging in and seeing that the right-hand side is maximal when
\( t = 0 \), we get:
\[
d_{\text{max}} = \frac{\sigma}{C} = \frac{15,000 \text{ bits}}{300 \text{ kbps}} = 50 \text{ msec}
\]
(b) Verify that \( P > C > \rho \). With this, we can apply the formula
derived in class:
\[
b_{\text{max}} = \frac{P - C}{P - \rho} \cdot \sigma
\]
Plugging in values gives the result:
\[
b_{\text{max}} = \frac{0.7 \text{ Mbps}}{0.85 \text{ Mbps}} \cdot 15,000 \text{ bits} \approx 12353 \text{ bits}.
\]
From the class notes, you see that for dual-leaky buckets at a
constant rate link, we have \( d_{\text{max}} = b_{\text{max}}/C \). Plugging values in
gives:
\[
d_{\text{max}} = \frac{b_{\text{max}}}{C} = \frac{12353 \text{ bits}}{300 \text{ kbps}} \approx 41.17 \text{ msec}.
\]
(c) The solution is a simple application of the network service curve, i.e., the service curve given by the network as a whole is the convolution of the service curves of each node. You need to realize that the convolution for constant-rate service curves is the smallest rate:

\[(C_1 \cdot t^+) \ast (C_2 \cdot t^+) \ast (C_3 \cdot t^+) = \min \{C_1, C_2, C_3\} \cdot t^+\]

Hence, the network gives a rate of \(C_{net} = 200 \text{kbps}\). Plugging in values for the formulas obtained in part (b), we get:

\[b_{\text{max}} = \frac{0.8 \text{ Mbps}}{0.85 \text{ Mbps}} \cdot 15,000 \text{bits} \approx 14117.6 \text{bits} .\]

\[d_{\text{max}} = \frac{b_{\text{max}}}{C_{\text{net}}} = \frac{b_{\text{max}} \text{bits}}{200 \text{kbps}} \approx 70.6 \text{msec} .\]