ECE 466 - Computer Networks II

Problem Set #1

This problems set contains exercises with the convolution operator '$\otimes$', defined as follows.

Definition: For two functions f and g the (min-plus) convolution is defined by:

$$f \otimes g(t) = \inf_{0 \le s \le t} \{f(s) + g(t-s)\}$$

Problem 1. Show that for two non-decreasing functions that satisfy $f(t) = g(t) = 0$ for $t \le 0$, the following holds:

$$f \otimes g(t) = \inf_{0 \le s \le t} \{f(s) + g(t-s)\} = \inf_{s \in \mathbb{R}} \{f(s) + g(t-s)\} \ .$$

Note: A function f for which $f(t) = 0$ if $t < 0$ is called one-sided or *causal*.

Solution. With the assumptions, we can show that the infimum is attained in the range $0 \le s \le t$. Suppose that the infimum is attained at $s^* < 0$, i.e., $f \otimes g(t) = f(s^*) + g(t - s^*)$. However, this is a contradiction since $f(s^*) = f(0)$ and $g(t - s^*) \ge g(t)$ (i.e., we do not increase the sum by setting $s = 0$).

Likewise, suppose that the infimum is attained at $s^* > t$, i.e., $f \otimes g(t) = f(s^*) + g(t - s^*)$. This also yields a contradiction since $f(s^*) \ge f(t)$ and $g(t - s^*) = g(0)$ (i.e., we do not increase the sum by setting $s = t$).

Problem 2. Given the functions

$$f(t) = \begin{cases} 0 & \text{if } t \le 0 \\ t+3 & \text{if } t > 0 \end{cases}$$

$$g(t) = \begin{cases} 0 & \text{if } t \le 0 \\ 2t+1 & \text{if } t \ 0 \end{cases}$$

Compute the *convolution* of $f \otimes g$.

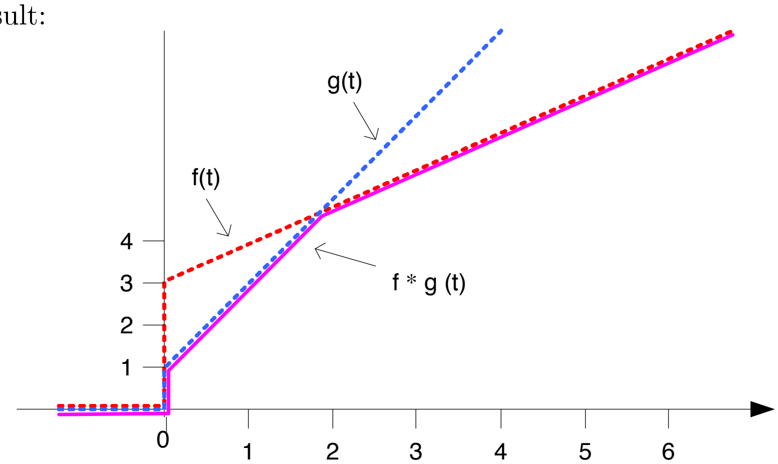
Solution. We will write the functions as :

$$f(t) = (t+3) \cdot I_{t>0}$$
$$g(t) = (2t+1) \cdot I_{t>0}$$

where $I_{t>0}$ is the indicator function with:

$$I_{t>0} = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases}$$

To get an idea of the solution, first try to visualize the solution. Here is the result:



Compare the result above, with the following rule: *Rule:* If f and g are both concave functions, the $f \otimes g \leq \min(f, g)$.

The functions and the result of the convolution $f \otimes g$ is shown in the figure. We distinguish two cases: (1) $t \leq 0$ and (2) $t > 0$.

First we consider the case $t \leq 0$. From the definition of the convolution it follows that for two causal processes processes where $f(t) = 0$ or $g(t) = 0$, we have $f \otimes g(t) = 0$ (see Problem 1). So we have,

$$f \otimes g(t) = 0 \;, \quad \text{if } t \leq 0.$$

The remainder deals with the case $t > 0$. Let us write:

$$f \otimes g(t) = \inf_{0 \leq s \leq t} \{(s+3) \cdot I_{s>0} + (2(t-s)+1) \cdot I_{s<t}\}$$

Because of the indicator functions, we have that $f(s) + g(t-s)$ as a function of s looks different in the following three ranges:

- $s = 0$: In this range $f(s) = 0$, and $g(t - s) > 0$.
- $0 < s < t$: In this range both $f(s) > 0$ and $g(t - s) > 0$.
- $s = t$: In this range, $f(s) > 0$ and $g(t - s) = 0$.

So, we can split up the infimum into the three segments:

$$\begin{aligned} f \otimes g(t) &= \min\Big\{\inf_{s=0}\{2(t-s)+1\},\ \inf_{0<s<t}\{(s+3)+(2(t-s)+1)\},\ \inf_{s=t}\{s+3\}\Big\} \\ &= \min\big\{2t+1,\ t+4,\ t+3\big\} \\ &= \min\big\{2t+1,\ t+3\big\} \end{aligned}$$

In the second line, we have inserted the values for $s = 0$ and $s = t$ and used that

$$\inf_{0<s<t}\{(s+3)+(2(t-s)+1)\} = \inf_{0<s<t}\{2t-s+4\} = t+4 \ ,$$

where the infimum is reached for $s \to t$. Now we have the solution:

$$\begin{aligned} f \otimes g(t) &= \begin{cases} 0 & t \le 0 \\ \min(t+3, 2t+1) \ , & t > 0 \end{cases} \\ &= \min(t+3, 2t+1)\ I_{t>0} \ . \end{aligned}$$

Problem 3.

Given the functions

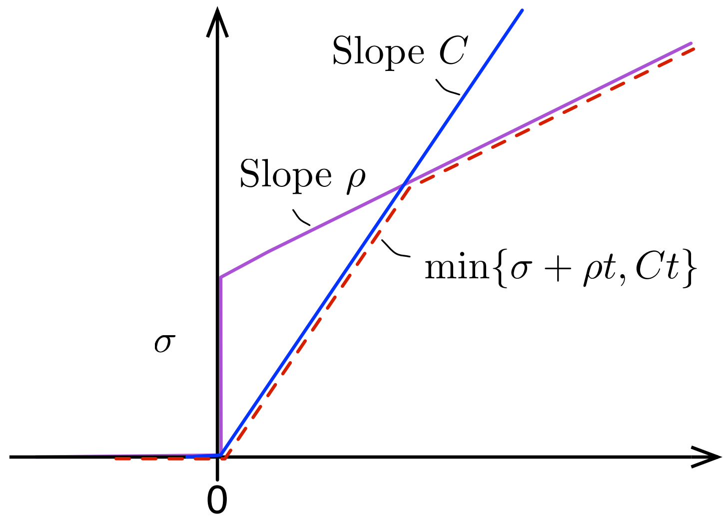
$$\begin{aligned} A(t) &= (\sigma + \rho t) I_{t>0} \\ S(t) &= (C \cdot t) I_{t>0} \end{aligned}$$

where σ, ρ and C are non-negative constants and $C > \rho$.

(a) Sketch the function $A \otimes S$.

(b) Use Reich's backlog equation to show that the queue length at any time t is bounded by σ.

Solution.



(a) Since $A(t) = S(t) = 0$ if $t \leq 0$. This gives us,

$$A \otimes S(t) = 0 \ , \quad \text{if } t \leq 0.$$

In the following we assume that $t > 0$. Let us first write the convolution results:

$$A \otimes S(t) = \inf_{s \in \mathbb{R}} \{A(s) + S(t-s)\}$$

Here, $A(s) + S(t-s)$ has a different behavior in each of the following ranges:

- $s \leq 0$: In this range $A(s) = 0$, and $S(t-s) > 0$.
- $0 < s < t$: In this range both $A(s) > 0$ and $S(t-s) > 0$.
- $s \geq t$: In this range, $A(s) > 0$ and $S(t-s) = 0$.

Splitting up the range of the infimum, we get

$$A \otimes S(t) = \min\Big\{\inf_{s \leq 0}\{C(t-s)\}, \inf_{0<s<t}\{(\sigma + \rho s) + C(t-s)\}, \inf_{s \geq t}\{\sigma + \rho s\}\Big\}$$

The first term is smallest when we select $s = 0$, the second term is minimized for $s \to t$, and the third term is smallest for $s = t$. The result is:

$$A \otimes S(t) = \min\Big\{Ct, \sigma + \rho t, \sigma + \rho t\Big\} = \min\Big\{Ct, \sigma + \rho t\Big\} \ .$$

Including the solution for $t \leq 0$, we get

$$A \otimes S(t) = \min\{Ct, \sigma + \rho t\} I_{t>0} \ .$$

(b) Since $B(0) = A(0) - D(0) = 0$ by definition, we only consider $t > 0$. The maximum backlog is given by

$$B^{\max} = \sup_{t>0} B(t) \, .$$

The backlog according to Reich's inequality is

$$B(t) = \sup_{0 \leq s \leq t} \{\sigma + \rho t - (\sigma + \rho s) I_{s>0} - C(t-s) I_{t>s}\} \, .$$

We break the computation of the supremum into three subintervals: $s = 0, 0 < s < t$, and $s = t$,

$$\begin{aligned} B(t) &= \max \left\{ \underbrace{\sigma + (\rho - C)t}_{s=0} \, , \ \sup_{0<s<t} \{(\rho - C)(t-s)\} \ , \ \underbrace{0}_{s=t} \right\} \\ &= \max\{\sigma + (\rho - C)t, 0, 0\} \\ &= \sigma + (\rho - C)t \ . \end{aligned}$$

We get $\sup_{0<s<t}\{(\rho - C)(t-s)\} = 0$, since $(\rho - C)(t-s)$ is increasing in s, which means that the supremum is reached for $s \to 0$. Then we compute the maximum backlog as

$$B^{\max} = \sup_{t>0}\{\sigma + (\rho - C)t\} = \sigma \, .$$

Since $(\rho - C)t$ is decreasing in t, the supremum is reached if we pick t as small as possible ($t \to 0$).