Problem Set #1

This problem set contains exercises with the convolution operator \( \otimes \), defined as follows.

**Definition:** For two traffic processes \( f \) and \( g \) the (min-plus) convolution is defined by:

\[
f \otimes g(t) = \inf_{0 \leq s \leq t} \{ f(s) + g(t - s) \}
\]

**Problem 1.** Show that for two non-decreasing functions that satisfy \( f(t) = g(t) = 0 \) for \( t \leq 0 \), the following holds:

\[
f \otimes g(t) = \inf_{0 \leq s \leq t} \{ f(s) + g(t - s) \} = \inf_{s \in \mathbb{R}} \{ f(s) + g(t - s) \}.
\]

Note: A function \( f \) for which \( f(t) = 0 \) if \( t < 0 \) is called one-sided or **causal**.

**Problem 2.** Given the functions \( S_1 \) and \( S_2 \) with

\[
S_1(t) = \begin{cases} 
0, & \text{if } t \leq 1, \\
t + 1, & \text{if } t > 1.
\end{cases} \quad S_2(t) = \begin{cases} 
0, & \text{if } t \leq 2, \\
3, & \text{if } t > 2.
\end{cases}
\]

Compute the min-plus convolution \( S_1 \otimes S_2 \).

**Problem 3.**

Given the functions

\[
A(t) = (\sigma + \rho t)I_{t>0} \\
S(t) = (C \cdot t)I_{t>0}
\]

where \( \sigma, \rho \) and \( C \) are non-negative constants and \( C > \rho \).

(a) Sketch the function \( A \otimes S \).

(b) Use Reich’s backlog equation to show that the queue length at any time \( t \) is bounded by \( \sigma \).