ECE - Computer Networks II

Problem Set #2

Problem 1. Prove the following properties of the min-plus convolution for one-sided non-decreasing processes F, G, and H:

(a) *Associativity:* $(F \otimes G) \otimes H = F \otimes (G \otimes H)$.

(b) *Commutativity:* $F \otimes G = G \otimes F$.

(c) *Boundedness.* $F \otimes G \leq F$. In particular, $F \otimes F \leq F$.

Solution.

(a) *Associativity:* The proof can be done for any type of process. We first expand and then rearrange the applications of the 'inf' operators, and then perform a substitution $v = u - s$.

$$\begin{aligned}
(F \otimes G) \otimes H(t) &= \inf_{u\in\mathbb{R}}\{\inf_{s\in\mathbb{R}}\{F(s) + G(u-s)\} + H(t-u)\} \\
&= \inf_{u\in\mathbb{R}}\{\inf_{s\in\mathbb{R}}\{F(s) + G(u-s) + H(t-u)\}\} \\
&= \inf_{s\in\mathbb{R}}\{F(s) + \inf_{u\in\mathbb{R}}\{G(u-s) + H(t-s-(u-s))\}\} \\
&= \inf_{s\in\mathbb{R}}\{F(s) + \inf_{v\in\mathbb{R}}\{G(v) + H(t-s-v)\}\} \\
&= \inf_{s\in\mathbb{R}}\{F(s) + G \otimes H(t-s)\} \\
&= F \otimes (G \otimes H)(t)
\end{aligned}$$

(b) *Commutativity:* For any three processes F, G and H, this can be shown by simply performing a substitution $x = t - s$.

$$\begin{aligned}
F \otimes G(t) &= \inf_{s\in\mathbb{R}}\{F(s) + G(t-s)\} \\
&= \inf_{t-x\in\mathbb{R}}\{F(t-x) + G(x)\} \\
&= \inf_{x\in\mathbb{R}}\{G(x) + F(t-x)\} \\
&= G \otimes F(t) \ ,
\end{aligned}$$

where we have used that '$t - x \in \mathbb{R}$' is equivalent to '$x \in \mathbb{R}$'.

(c) *Boundedness.* $F \otimes G \leq F$. In particular, $F \otimes F \leq F$.

If G is one-sided, then we have $G \leq \delta$. Then, for a non-decreasing process F, we have $F \otimes G \leq F \otimes \delta = F$. If F is both one-sided and non-decreasing, we can set $G = F$ and get $F \otimes F \leq F$.

Problem 2. Given the functions f and g, compute the convolution $f \otimes g$:

$$f(t) = \begin{cases} 0, & t \leq 1 \\ 2, & 1 < t \leq 2 \\ 2+t, & t > 2 \end{cases}, \qquad g(t) = \begin{cases} 0, & t \leq 2 \\ 1+2t & t > 2 \end{cases}$$

Solution.

First, we define $G'(t)$ as time shifted version of $G(t)$:

$$G'(t) = \begin{cases} 0, & t \leq 0 \\ 2t+5 & t > 0 \end{cases}$$

Since we have $G(t) = G' \otimes \delta_2(t)$ by the time shift property of the δ function, the convolution can be rewritten as $F \otimes G(t) = F \otimes G' \otimes \delta_2(t)$. Next, we rewrite $F(t)$ and $G'(t)$ using the indicator function:

$$F(t) = 2 \cdot I_{1<t\leq 2} + (2+t) I_{t>2}$$
$$G'(t) = (2t+5) I_{t>0}$$

First we solve the subproblem of $F \otimes G'(t)$. Later, we apply the time shift $\delta_2(t)$.

$$\begin{aligned} F \otimes G'(t) &= \inf_{0\leq s\leq t} \left\{ F(s) + G'(t-s) \right\} \\ &= \inf_{0\leq s\leq t} \{ 2 \cdot I_{1<s\leq 2} + (2+s) I_{s>2} + (2(t-s)+5)\, I_{t-s>0} \} \\ &= \min \left\{ \inf_{0\leq s\leq 1} \{ (2(t-s)+5)\, I_{s<t} \}, \inf_{1<s\leq 2} \{ 2 + (2(t-s)+5)\, I_{s<t} \}, \right. \\ &\qquad \left. \inf_{2<s\leq t} \{ 2 + s + (2(t-s)+5)\, I_{s<t} \} \right\} \end{aligned}$$

We next compute the infimums appearing in the above expression:

(1) $\inf_{0\le s\le 1}\{(2(t-s)+5)\,I_{s<t}\}$
First, observe that if the indicator function $I_{s<t}$ evaluates to one, that is, $s<t$, we have $2(t-s)+5>0$. Therefore, the function $(2(t-s)+5)\,I_{s<t}$ achieves the minimum value of zero when $s\ge t$ and the indicator function $I_{s<t}$ evaluates to zero. Since s is in the range $0\le s\le 1$, we can select a value of s that satisfies $s\ge t$ if we have $t<1$.
In the second case, when $t\ge 1$, the indicator function evaluates to one and we perform the minimization $\inf_{0\le s\le 1}\{2(t-s)+5\}$. The minimum is obtained for $s=1$ and is equal to $2t+3$.
The result of the minimization is:

$$\begin{aligned}\inf_{0\le s\le 1}\{(2(t-s)+5)\,I_{s<t}\} &= \begin{cases}0, & t<1\\ 2t+3, & t\ge 1\end{cases}\\ &= (2t+3)I_{t\ge 1}\end{aligned}$$

(2) $\inf_{1<s\le 2}\{2+(2(t-s)+5)\,I_{s<t}\}$
If the indicator function $I_{s<t}$ evaluates to one, i.e., $s<t$, then $2+(2(t-s)+5)>2$. If the indicator function $I_{s<t}$ evaluates to zero, then the value of the function is equal to 2. Thus, we obtain the minimum of 2 when $s\ge t$, which we can achieve if $t<2$. In the case when $t\ge 2$ we minimize $\inf_{1<s\le 2}\{2+2(t-s)+5\}$. The minimum is obtained for $s=2$ and is equal to $2t+3$.
The result of the minimization is:

$$\begin{aligned}\inf_{1<s\le 2}\{2+(2(t-s)+5)\,I_{s<t}\} &= \begin{cases}2, & t<2\\ 2t+3, & t\ge 2\end{cases}\\ &= \begin{cases}2, & t<2\\ 2+(2t+1), & t\ge 2\end{cases}\\ &= 2+(2t+1)I_{t\ge 2}\end{aligned}$$

(3) $\inf_{2<s\le t}\{2+s+(2(t-s)+5)\,I_{s<t}\}$
If the indicator function $I_{s<t}$ evaluates to zero (for $s=t$) then we have $\inf_{s=t}\{2+s\}=2+t$. If the indicator function $I_{s<t}$ evaluates to one (for $s<t$) then we have $\inf_{2<s<t}\{2+s+2(t-s)+5\}=\inf_{2<s<t}\{2t-s+7\}$, which is minimized for $s\to t$ and is equal to $t+7$.
Since $t+2<t+7$, the minimum is obtained for $s=t$ when the indicator function $I_{s<t}$ evaluates to zero. The minimum is

$$\inf_{2<s\le t}\{2+s+(2(t-s)+5)\,I_{s<t}\} = t+2$$

Applying the computed minimums to the convolution we obtain

$$f \otimes G'(t) = \min\{(2t+3)\, I_{t\geq 1}, 2 + (2t+1)\, I_{t\geq 2}, t+2\}$$

$$= \begin{cases} 0, & t < 1 \\ 2, & 1 \leq t < 2 \\ t+2, & t \geq 2 \end{cases}$$

Finally we apply the time shift

$$F \otimes G(t) = F \otimes G' \otimes \delta_2(t)$$

$$= \begin{cases} 0, & t < 3 \\ 2, & 3 \leq t < 4 \\ t, & t \geq 4 \end{cases}$$

Problem 3. Define the function $f_n(t) = nI_{t>0}$ for a number $n \in \mathbb{N}$.

1. Show that the function is a traffic process, i.e., f_n is one-sided and non-decreasing.
2. Show the result of the convolution $f_3 \otimes f_2$.

Solution.

A function is one-sided if $f(t) = 0$ for $t \leq 0$. Function $f_n(t) = 0$ for $t \leq 0$ as indicator function $I_{t>0}$ evaluates to zero and thus is one-sided. A function is non-decreasing if $f(t+s) \geq f(t)$ for all $s > 0$.

$$f_n(t+s) = nI_{t+s>0} \geq nI_{t>0} = f_n(t)$$

For $t > 0$ both indicator functions evaluate to 1 (as $s > 0$) and functions are equal. For $t < 0$, if $s \leq -t$ both indicator functions evaluate to 0 making functions equal, and if $s > -t$ first indicator function evaluates to 1 and second to 0 making $f_n(t+s) = n \geq f_n(t) = 0$ as $n \in \mathbb{N}$.

And the convolution is:

$$f_3 \otimes f_2(t) = \inf_{0\leq s\leq t} \{f_3(s) + f_2(t-s)\}$$

$$= \inf_{0\leq s\leq t} \{3I_{s>0} + 2I_{s<t}\}$$

$$= min\left\{\inf_{s=0}\{2\}, \inf_{0<s<t}\{3+2\}, \inf_{s=t}\{3\}\right\}$$

$$= 2I_{t>0}$$

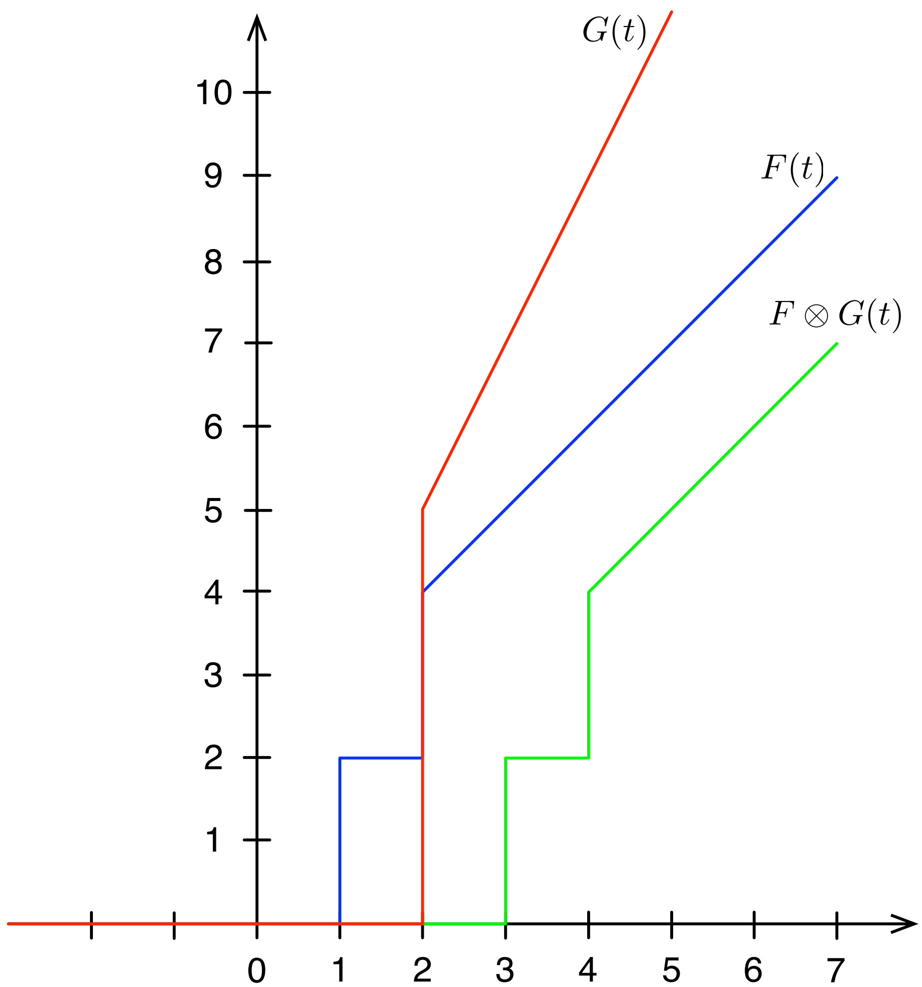

Figure 1: Convolution $F \otimes G(t)$.