Homework #6

1. Papoulis & Pillai 15-4
2. Papoulis & Pillai 15-5
3. Papoulis & Pillai 15-6
4. (*ALG 8-15*) A critical part of a machine has an exponentially distributed lifetime with parameter α. Suppose that n spare parts are initially in stock, and let $N(t)$ be the number of spares left at time t.

   a. Find $p_{ij}(t)$.

   b. Find the transition probability matrix.

   c. Find $p_j(t)$.

5. (Resnick, *Adventures in Stochastic Processes - The Random World of Happy Harry*) **Harry Meets Sleeping Beauty**. Harry dreams he is Prince Charming coming to rescue Sleeping Beauty (SB) from her slumbering imprisonment with a kiss. The situation is more complicated than in the original tale, however, as SB sleeps in one of three positions:

   (1) flat on her back, in which case she looks truly radiant;

   (2) fetal position, in which case she looks less than radiant;

   (3) fetal position and sucking her thumb in which case she looks radiant only to an orthodontist.

   SB's changes of position occur according to a Markov chain with transition matrix

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0 & 0.75 & 0.25 \\ 0.25 & 0 & 0.75 \\ 0.25 & 0.75 & 0 \end{pmatrix} .$$

   SB stays in each position for an exponential amount of time with parameter $\lambda(i), 1 \le i \le 3$, measured in hours, where

$$\lambda(1) = 1/2, \lambda(2) = 1/3, \lambda(3) = 1 .$$

   Assume for the first two questions that SB starts sleeping in the truly radiant position.

   (a) What is the long run percentage of time SB looks truly radiant?

   (b) If Harry arrives after an exponential length of time (parameter α), what is the probability he finds SB looking truly radiant? (Try Laplace transform and matrix techniques.) (*The only solution I can find requires access to Maple or Mathematica. - BL*)

(c) SB, being a delicate princess, gets bed sores if she stays in any one position for too long, namely if she stays in any position longer than three hours. Define for $t > 0$ and $i = 1, 2, 3$,

$$S_i(t) = Pr\{\text{no bed sores up to time } t | \text{SB's initial position is } i\} \ ,$$

so that $S_i(t) = 1$ for $t \leq 3$. Write a recursive system of equations satisfied by the functions $S_i(t), t > 0, i = 1, 2, 3$. You do not have to solve this system.

6. Papoulis $\&$ Pillai 16-1. Typo: r should be m.

7. Papoulis $\&$ Pillai 16-3

8. Papoulis $\&$ Pillai 16-5

9. (Prabhu, *Foundations of Queueing Theory*) A gas station has room for seven cars including the ones at the pumps. The installation of a pump costs $\$50$ per week and the average profit on a customer is 40 cents. Customers arrive in a Poisson process at a rate of 4 per minute, and the service times have exponential density with mean 1 minute. Find the number of pumps which will maximize the expected net profit.

10. (*ALG 8-26*) N balls are distributed in two urns. At time n, a ball is selected at random, removed from its present urn, and placed in the other urn. Let X_n denote the number of balls in urn 1.

    a. Find the transition probabilities for X_n.

    b. Argue that the process is time-reversible and then obtain the steady state probabilities for X_n.