Unicast Multi-antenna Relay Beamforming with Per-Antenna Power Control: Optimization and Duality

Min Dong, *Senior Member, IEEE*, Ben Liang, *Senior Member, IEEE*, and Qiang Xiao

***Abstract*—We consider amplify-and-forward multi-antenna relaying between a single pair of source and destination under relay per-antenna power constraints. We design the optimal relay processing matrix to minimize the maximum per-antenna power budget for a received SNR target. With given transmit and receive beamformers at the source and destination, respectively, we first focus on the equivalent system with single-antenna source and destination. Although non-convex, we show that the optimization satisfies strong Lagrange duality and can be solved in the Lagrangian dual domain. We reveal a prominent structure of this problem, by establishing its duality with direct SIMO beamforming system with an uncertain noise. This enables us to derive a semi-closed form expression for the optimal relay processing matrix that depends on a set of dual variables, which can be determined through numerical optimization with a significantly reduced problem space. We further show that the dual problem has a semi-definite programming form, which enables efficient numerical optimization methods to determine the dual variables with polynomial complexity. Using this result, the reverse problem of SNR maximization under a set of relay per-antenna power constraints is then addressed. We then consider the maximum relay beamforming achievable rate under different combinations of antenna setups at source and destination. In particular, we generalize the duality to MIMO relay beamforming vs. direct MIMO beamforming, and establish the dual relation of the two systems for different multi-antenna setups at source and destination.**

***Index Terms*—multi-antenna relaying, amplify-and-forward, relay beamforming, per-antenna power, achievable rate, Lagrange duality,**

I. Introduction

We study the optimal design of multi-antenna relay processing in amplify-and-forward (AF) multiple-input multiple-output (MIMO) relaying systems. With multiple antennas equipped at the relay, a processing matrix is used to linearly process the received signals and forward them to the destination. We specifically consider the relay beamforming problem, where the processing matrix is designed to maximize the destination received signal-to-noise (SNR) ratio. The central design problem is finding the optimal relay processing matrix. It often involves finding both the structure of the optimal processing matrix and the jointly optimal power allocation.

For transmission between a single pair of source and destination, an optimal design of the processing matrix has been studied under different performance criteria, such as capacity, diversity gain, SNR maximization, and relay power minimization [2]–[6]. For many cases studied, the processing matrix inherits a beamforming structure characterized by the channels at the first and second hops. The relay processing design for multiple sources and/or destinations has also been studied in [7]–[9]. The explicit solution for the optimal processing matrix is difficult to obtain in such setups. Either numerical methods are proposed to obtain approximate solution for the optimal processing matrix, or suboptimal structure is imposed to simplify the problem. Regardless of single or multiple pairs of source and destinations, all these existing results rely on the sum-power constraint across antennas at the relay. In general, the sum-power constraint leads to more analytically tractable problems, allowing certain system structure to be explored in obtaining the solution.

In a practical system, however, the implementation of multi-antenna relaying imposes different power constraints. Each antenna is limited by its own RF front-end power amplifier, so that a realistic multi-antenna relay processing design is constrained by a per-antenna power budget[1]. For multiple relays each equipped with a single antenna to collaboratively form a virtual multi-antenna system for cooperative communications, individual antenna power budget is particularly more realistic. With such per-antenna power constraints, the relay processing design optimization becomes more challenging. For the case of a single pair of source and destination, none of the approaches developed in [2]–[6] is applicable to solve the problem. To our best knowledge, no previous related results have been reported.

Aside from the optimal relay processing design, it is also important to understand the relation between MIMO relay beamforming systems and direct MIMO beamforming systems. Whether and in what sense, a MIMO relay system can be equivalently viewed as some direct MIMO transmission system. In other words, whether there exists a duality of the two types of systems. Most existing results focus on the design of the optimal relay processing matrix but do not provide insights of such relation.

Our objective in this work is to obtain the solution for the

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[1]Although per-antenna power amplifier is typical in most systems, there has been some recent progress in microwave and antenna design to provide central power supply to multiple antennas from an amplifier network, which could be used for certain communication systems, such as satellite communications [10].

optimal relay processing design in MIMO relay beamforming with per-antenna power control, and investigate the duality relation of such relay systems with direct MIMO systems for both processing design and maximum achievable rate. Our approach is inspired by the framework in [11] for direct downlink multi-antenna transmission, where the optimal transmit beamforming design is obtained under per-antenna power constraints. However, different from downlink beamforming, multi-antenna relaying structure leads to a unique structure for the received SNR, which depends on the channels over two hops and the inherent noise amplification, in addition to the per-antenna power control at the relay. This complicates the optimization problem with new challenges.

A. Contributions

We consider the relay processing optimization with per-antenna power control for unicast dual-hop AF MIMO relaying system with a single data stream. To design the optimal relay processing matrix, we first cast the problem as a power minimization problem to minimize the maximum power consumption among the relay antennas.

Through reformulation, we transform the originally non-convex problem into an equivalent problem which is shown to have zero duality gap. Interestingly, through the Lagrange dual method, we establish a duality between multi-antenna relay beamforming system and direct single-input multiple-output (SIMO) beamforming system with a dual SIMO channel formed by concatenating the two-hop relay channels, and uncertain noise covariance. This enables us to derive a semi-closed form expression for the optimal relay processing matrix. The semi-closed form expression is parameterized by the Lagrangian dual variables to be determined numerically. With N relay antennas, this solution not only presents the structure of the optimal processing matrix, but also allows us to convert the original optimization problem with N^2 variables and $(N+1)$ constraints, to one with $(N+1)$ variables and three constraints. To determine the dual variables, we further show that the dual problem has a semi-definite programming (SDP) form, which can be efficiently solved using interior-point methods with polynomial complexity [12]. This greatly reduces the computation complexity in determining the final solution. The solution applies to the case of single-antenna source and destination, or multi-antenna source and destination with given transmit and beamforming vectors. Discussion of joint design of the relay processing matrix, and the source/desitnation beamforming vectors is also provided.

Following the power minimization problem, we further consider the reverse problem of SNR maximization with given per-antenna power constraints. We show that the two problems are inverse problems with monotonic relation of SNR and power constraints, thus the solution to the SNR problem can be obtained through iteratively solving the power minimization problem along with bisection search. The optimal relay processing solution obtained enables us not only to compare the performance difference under per-antenna power and sum-power constraints, but also to evaluate the performance gap between the centralized and distributed relay beamforming systems, under per-antenna/per-node power constraints, to more accurately quantify the loss due to distributed processing.

We next investigate the maximum achievable rate of the MIMO relay beamforming system with relay per-antenna power constraints. For source and destination equipped with a single antenna, the duality established earlier indicates that the maximum achievable rates of the multi-antenna relay system and direct SIMO beamforming system are identical. For source with single antenna and destination with multiple antennas, or vice versa, we show that the dual of the relay system is a MIMO system with a dual MIMO channel structured from the two-hop relay channels and uncertain noise covariance; the maximum achievable rate of such relay system is identical to the maximum beamforming achievable rate of the dual MIMO system. When source and destination are both equipped with multiple antennas, we show that the dual relation of the MIMO relay system to a direct MIMO system still holds for beamforming, but the maximum beamforming achievable rate in the relay system is upper bounded by that of the direct MIMO system, and the two may not be guaranteed to be identical.

B. Related Work

The relay processing design in multi-antenna AF relaying systems has drawn considerable attention in recent years (see [13] and references therein). Among the existing results, the sum-power constraint at the relay (or a total power constraint at the source and the relay) is typically assumed. Under this assumption, for the purpose of relay beamforming to maximize the received SNR, with a single pair of source and destination, the optimal design of the processing matrix was given in [4], where a rank-one beamforming matrix structure was found. The result was further extended to the case when only the second-order statistics of the relay channels are known at the relay [5]. For multiple pairs of sources and destinations, the closed-form solution for multi-antenna relay processing matrix under the sum-power constraint is difficult to obtain, and either numerical algorithms [9] or suboptimal approaches [7], [8] were proposed. The MMSE-based criteria was also used in designing the processing matrix [6], [14], [15]. For the purpose of maximizing the MIMO AF relay capacity, the optimal processing matrix was sought for a multi-antenna relay under the relay sum-power constraint [2], [3], [16]–[18]. Besides dual-hop relaying, the design of processing matrix for multi-hop AF MIMO relaying was considered for total power minimization [19]. In addition to one-way relaying systems, the design of relay processing matrix for two-way multi-antenna AF relaying systems was investigated [20], [21].

Despite all these results, the investigation of multi-antenna relay processing design under per-antenna power constraints is scarce, and the problem has remained open. For direct point-to-point systems, the optimal transmit beamforming design under per-antenna power constraints was obtained in [11], and the uplink-downlink duality for the beamforming SINR region as well as the capacity region was obtained.

For AF MIMO relaying systems with the sum-power constraint at each relay, the uplink-downlink duality for the achievable rate region was established for dual-hop transmission [22] and was extended to the multi-hop scenario [23]. Note that the duality obtained there focused on a different system equivalence from the duality established in this work. There, the comparison is between uplink and downlink transmission through relaying. In our work, the comparison is between two types of transmission systems, i.e. the relaying system and the direct transmission system.

C. Organization and Notations

The rest of the paper is organized as follows. In Section II, we present the system model and problem formulation. In Section III, we provide the optimal design solution and establish the duality between the relay beamforming system with per-antenna power control and the direct SIMO beamforming system with an uncertain noise covariance. We also discuss the reverse problem of SNR maximization under per-antenna constraints and its relation to the power minimization problem. In Section IV, we consider the maximum achievable rate of the relay beamforming system with per-antenna power constraints, and establish the duality relation of the relay system with certain direct MIMO systems under different source and/or destination antenna setups. We present numerical results in Section V and conclude this work in Section VI.

Notations: $\|\cdot\|$ denotes the Euclidean norm of a vector, and $\otimes$ stands for the Kronecker product. Hermitian and transpose are denoted as $(\cdot)^H$ and $(\cdot)^T$, respectively. Conjugate is denoted as $(\cdot)^*$. Matrix pseudo-inverse is denoted as $(\cdot)^\dagger$. We use $[\mathbf{a}]_i$ to denote the ith element of vector $\mathbf{a}$; and $[\mathbf{A}]_{ii}$ to denote the ith diagonal entry of matrix $\mathbf{A}$. The notation $\mathbf{A} \succcurlyeq (\preccurlyeq) \mathbf{B}$ means that the matrix $(\mathbf{A}-\mathbf{B})$ is positive (negative) semi-definite, while $\mathbf{a} \succcurlyeq 0$ means that the vector $\mathbf{a}$ is element-wise non-negative. The notation $\mathrm{vec}(\mathbf{A})$ vectorizes the matrix $\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_N]$ to $[\mathbf{a}_1^T, \cdots, \mathbf{a}_N^T]^T$. The notation $\mathcal{CN}(m, \sigma^2)$ denotes proper complex Gaussian distribution with mean m and variance σ^2.

II. Problem Formulation

A. System Model

We consider a unicast dual-hop MIMO AF relaying system where a pair of source and destination nodes, equipped with M_s and M_d antennas respectively, communicate through a relay equipped with N antennas, as illustrated in Fig. 1.The direct link is ignored. We consider the system transmitting a single-data stream through source-destination beamforming.[2]

The data forwarding takes place in two phases. In the first phase, the source transmits the signal to the relay. The received signal vector at the relay is given by $\mathbf{y_r} = \mathbf{H}_1\mathbf{b}\sqrt{P_o}s + \mathbf{n_r}$, where s is the transmitted signal from the source with unit power $\mathrm{E}|s|^2 = 1$, P_o is the total transmit power at the source, $\mathbf{b}$ is a $M_s \times 1$ unit-norm transmit beamforming vector, $\mathbf{H}_1$ is the $N \times M_s$ complex channel matrix between the source and the relay, and $\mathbf{n}_r$ is the $N \times 1$ complex additive white Gaussian

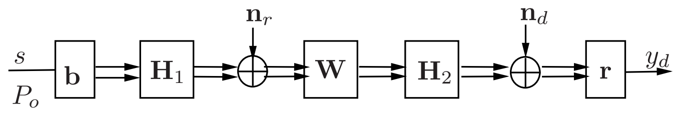


Fig. 1: An AF MIMO relaying system.

noise (AWGN) vector with covariance $\sigma_r^2\mathbf{I}$, where $\mathbf{I}$ is an $N \times N$ identity matrix[3]. In the second phase, the received signals at the relay are processed with an $N \times N$ relay processing matrix $\mathbf{W}$ and then are forwarded to the destination. The received signal at the destination, after receive beamforming, is given by

$$y_d = \mathbf{r}^H\mathbf{H}_2\mathbf{W}\mathbf{H}_1\mathbf{b}\sqrt{P_o}s + \mathbf{r}^H\mathbf{H}_2\mathbf{W}\mathbf{n_r} + \mathbf{r}^H\mathbf{n}_d \qquad (1)$$

where $\mathbf{H}_2$ is the $M_d \times N$ complex channel matrix between the relay and the destination, $\mathbf{r}$ is a $M_d \times 1$ unit-norm receive beamforming vector $\|\mathbf{r}\|^2 = 1$, and $\mathbf{n}_d$ is the $M_d \times 1$ AWGN vector at the destination receiver with with i.i.d. elements, each with variance σ_d^2.

For given $\mathbf{b}$ and $\mathbf{r}$, the system can be equivalent to one with single-antenna source and destination given by

$$y_d = \mathbf{h}_2^T\mathbf{W}\mathbf{h}_1\sqrt{P_o}s + \mathbf{h}_2^T\mathbf{W}\mathbf{n}_r + n_d \qquad (2)$$

with the equivalent channel vector at the 1st and 2nd hops as

$$\mathbf{h}_1 \triangleq \mathbf{H}_1\mathbf{b}, \quad \mathbf{h}_2 \triangleq (\mathbf{H}_2^H\mathbf{r})^*, \quad n_d \triangleq \mathbf{r}^H\mathbf{n}_d \qquad (3)$$

and n_d is AWGN with variance σ_d^2. We assume perfect knowledge of $\mathbf{h}_1$ and $\mathbf{h}_2$ at the relay.

In Section III, we will focus on optimally designing the relay processing matrix $\mathbf{W}$, with given source and destination beamforming vectors $\mathbf{b}$ and $\mathbf{r}$. Thus, with no loss of generality, we directly consider the equivalent system in (2) and (3). The joint design of $\mathbf{W}$, $\mathbf{b}$, and $\mathbf{r}$ is discussed in Section III-G. The MIMO relay beamforming maximum achievable rates, among all possible $(\mathbf{b}, \mathbf{r})$ pairs, under different source/destination antenna configurations are detailed in Section IV.

B. Relay Processing with Per-Antenna Power Control

By (2), the received signal-to-noise ratio (SNR) at the destination is obtained as

$$\mathrm{SNR} = \frac{P_o|\mathbf{h}_2^T\mathbf{W}\mathbf{h}_1|^2}{\sigma_r^2\|\mathbf{h}_2^T\mathbf{W}\|^2 + \sigma_d^2}. \qquad (4)$$

Various end-to-end performance measures, such as data rate or bit-error-rate (BER), are direct functions of the received SNR given above. System optimization under these performance metrics can then be directly converted to the optimization problem under the SNR metric. Thus, in the following, we focus on the SNR metric.

With the practical assumption that each transmit antenna at the relay is individually power controlled with its own power budget, our objective is to design an optimal $\mathbf{W}$ at the relay to minimize the per-antenna power usage for data forwarding,

[2] In the case of multiple transmit antennas at the source ($M_s > 1$), the single-stream MIMO beamforming structure can be used for a system seeking maximum diversity gain and power gain to minimize data error probability.

[3] Throughout the paper, unless explicitly specified, $\mathbf{I}$ indicates an identity matrix with size $N \times N$.

subject to a given received SNR target. The per-antenna power constraint on the output of each transmit antenna at the relay is given by

$$\mathrm{E}\{|[\mathbf{W}\mathbf{y_r}]_i|^2\} = \left[P_0\mathbf{W}\mathbf{h}_1\mathbf{h}_1^H\mathbf{W}^H + \sigma_r^2\mathbf{W}\mathbf{W}^H\right]_{ii} \leq P_i \quad (5)$$

for $i = 1, \cdots, N$, where P_i is the power budget at the ith antenna. Note that we only consider the non-degenerate case where the forwarding link from each antenna of the relay to the destination is active, or equivalently, $|h_{2i}| > 0$, $\forall i$, where h_{2i} is the ith element in $\mathbf{h}_2$.[4] To formulate such per-antenna power minimization problem, we consider minimizing the maximum transmit power of each antenna at the relay for a given SNR target γ_o at the destination. This min-max optimization problem can be formulated as

$$\min_{\mathbf{W}} \max_{1\leq i\leq N} \; P_i \quad (6)$$

$$\text{subject to} \quad \mathrm{SNR} \geq \gamma_o, \quad (7)$$

$$\left[P_o\mathbf{W}\mathbf{h}_1\mathbf{h}_1^H\mathbf{W}^H + \sigma_r^2\mathbf{W}\mathbf{W}^H\right]_{ii} \leq P_i, \quad (8)$$

$$\text{for } i = 1, \cdots, N.$$

It is straightforward to show that the above min-max power minimization problem is equivalent to the problem of minimizing a common per-antenna power budget P_r, given by

$$\min_{\mathbf{W}} \; P_r \quad (9)$$

$$\text{subject to} \quad (7) \text{ and}$$

$$\left[P_o\mathbf{W}\mathbf{h}_1\mathbf{h}_1^H\mathbf{W}^H + \sigma_r^2\mathbf{W}\mathbf{W}^H\right]_{ii} \leq P_r, \quad (10)$$

$$\text{for } i = 1, \cdots, N.$$

The above problem also corresponds to a common practical scenario where identical hardware and front-end is used for each antenna.[5]

Besides the above power minimization problem, the reverse problem of SNR maximization under a set of relay per-antenna power constraints $\{P_1, \cdots, P_N\}$ is discussed in Section III-F.

III. Optimal Relay Beamforming Design

In this section, we provide the solution to the optimization problem (6). We first transform the received SNR expression in (4) through vectorizing the processing matrix $\mathbf{W}$. The reformulation enables the subsequent development of our results. Let $\mathbf{W}^H = [\mathbf{w}_1, \cdots \mathbf{w}_N]$. We have the following lemma.

Lemma 1: The received SNR expression in (4) for multi-antenna relay beamforming can be re-expressed in the following form

$$\mathrm{SNR} = \frac{P_o|\mathbf{h}^H\mathbf{w}|^2}{\left\|\mathbf{R}_{\mathbf{g}}^{\frac{1}{2}}\mathbf{w}\right\|^2 + \sigma_d^2} \quad (11)$$

where $\mathbf{w} \triangleq \mathrm{vec}(\mathbf{W}^H)$, $\mathbf{h} \triangleq \mathrm{vec}(\mathbf{h}_1\mathbf{h}_2^T) = \mathbf{h}_2 \otimes \mathbf{h}_1$, and $\mathbf{R_g} \triangleq (\mathbf{h}_2\mathbf{h}_2^H) \otimes \sigma_r^2\mathbf{I}$

Proof: See Appendix A.

In the following, we first provide the feasibility condition for the optimization problem (6). Then, we show how the problem can be transformed into a formulation, for which the Lagrange dual method can be applied to obtain the solution. The dual method leads to the establishment of the duality of multi-antenna relay beamforming to SIMO beamforming in direct point-to-point communication, leading to a semi-closed form solution for $\mathbf{W}$. Finally, we provide an SDP formulation as the numerical method to determine $\mathbf{W}$.

A. Feasibility Condition

The feasibility of the optimization problem (6) depends on the existence of $\mathbf{W}$ to satisfy the SNR constraint (7). It is determined by the values of the given transmit power P_o, the SNR target γ_o, and the relay channel conditions $\mathbf{h}_1, \mathbf{h}_2$. A feasibility condition for (6) is given as follow.

Proposition 1: A necessary condition for the multi-antenna relay beamforming problem (6) to be feasible is that the source transmit power P_o and destination SNR target γ_o satisfy

$$\frac{P_o\|\mathbf{h}_1\|^2}{\gamma_o\sigma_r^2} > 1. \quad (12)$$

Proof: See Appendix B. ■

Note that (12) is a necessary condition for the optimization (6) to exist. In Section III-D, a necessary and sufficient condition is given to guarantee the existence of the solution. However, that condition can only be verified through the optimization procedure we develop. Instead, the condition (12) can be verified before solving the problem.[6]

B. Strong Lagrange Duality

The optimization problem (6), and its equivalence (9), is non-convex due to the non-convex SNR constraint in (7) with respect to $\mathbf{W}$. Nonetheless, we show that the optimization can be solved in the Lagrange dual domain. Due to the equivalence of the optimization problems (6) and (9), in the following, we focus on the problem (9) instead.

Proposition 2: The optimization problem (9) has zero duality gap.

Proof: We provide a sketch of the proof and leave details in Appendix C. We first show that the optimization problem (9) can be converted to a second-order cone programming (SOCP) problem. Results in literature show that strong duality holds for the SOCP problem to its Lagrange dual problem. Thus, to complete the proof, we are left to show the Lagrange dual of (9) is equivalent to the Lagrangian dual of the SOCP problem. ■

As shown in the proof of Proposition 2, the optimization problem (9) can be converted to an SOCP problem, and

[4]The power constraint (5) is only meaningful for an active link. Otherwise, it is trivial to see that the antenna is inactive (*i.e.*, zero power) for data forwarding, and effectively the model is reduced to the one with only active links for relaying.

[5]Although we assumes no direct link, the formulation applies to the scenario when the direct link is available and is used in the first phase only. In this case, the combined received SNR at the destination is the summation of SNRs from the relay and the direct link, where the latter is not a function of $\mathbf{W}$. The problem would be the same as the one presented in (6).

[6]The condition (12) states that the received SNR on the first hop should be larger than the target SNR. Although quite intuitive, it is derived from the problem (6) with per-antenna power constraints. The proof in certain way indicates that the necessary condition is "tight". In our simulation studies, a majority of the infeasible cases can be eliminated by examining this necessary condition.

solved numerically using any standard SOCP package such as SeDuMi [24]. However, such numerical method will not provide any insight on the structure of the solution for $\mathbf{w}$. In addition, the optimization involves $(N+1)$ constraints and N^2 variables to be optimized. Instead, using Proposition 2, we next present the solution to the optimization problem (9) through its Lagrange dual. Through this approach, we eventually obtain the structure of the optimal $\mathbf{w}$ (or $\mathbf{W}$) and a more computationally efficient method to determine the optimal value of $\mathbf{w}$.

C. Duality with Point-to-Point SIMO Beamforming

Using the vectorized beamforming matrix, *i.e.*, $\mathbf{w} = [\mathbf{w}_1^H, \cdots, \mathbf{w}_N^H]^H$, and (53) in the proof of Proposition 2, the Lagrangian for (9) is given by

$$\begin{aligned} L(P_r, \mathbf{w}, \mathbf{\Lambda}, \nu) &= P_r + \nu \left\{ \sigma_d^2 + \|\mathbf{R}_g^{\frac{1}{2}} \mathbf{w}\|^2 - \frac{P_o}{\gamma_o} \left| \mathbf{h}^H \mathbf{w} \right|^2 \right\} \\ &+ \sum_{i=1}^{N} \lambda_i \left\{ \mathbf{w}_i^H \left(P_o \mathbf{h}_1 \mathbf{h}_1^H + \sigma_r^2 \mathbf{I} \right) \mathbf{w}_i - P_r \right\} \\ &= P_r + \nu \left\{ \sigma_d^2 + \|\mathbf{R}_g^{\frac{1}{2}} \mathbf{w}\|^2 - \frac{P_o}{\gamma_o} \left| \mathbf{h}^H \mathbf{w} \right|^2 \right\} \\ &+ \mathbf{w}^H \left[\mathbf{\Lambda} \otimes (P_o \mathbf{h}_1 \mathbf{h}_1^H + \sigma_r^2 \mathbf{I}) \right] \mathbf{w} - P_r \mathrm{tr}(\mathbf{\Lambda}) \end{aligned} \tag{13}$$

where $\mathbf{\Lambda} \triangleq \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ is the diagonal matrix of Lagrange multipliers corresponding to the per-antenna power constraints, and ν is the Lagrange multiplier corresponding to the received SNR target. The dual problem of (9) is given by

$$\max_{\mathbf{\Lambda}, \nu} \min_{P_r, \mathbf{w}} L(P_r, \mathbf{w}, \mathbf{\Lambda}, \nu) \tag{14}$$

$$\text{subject to } \mathbf{\Lambda} \succcurlyeq 0, \nu \geq 0. \tag{15}$$

To obtain a solution to the problem above, we first show that it can be transformed into the dual power minimization problem of SIMO beamforming in direct point-to-point communications.

For the SIMO beamforming problem under consideration, the transmitter has a single antenna with transmit power $\tilde{P}$. The receiver has N^2 antennas with a receiver noise covariance matrix $\tilde{\mathbf{\Sigma}}$. Assume that the channel is given by $\mathbf{h}$, and $\tilde{\mathbf{w}}$ is the receiver beamforming vector. The objective is to find the optimal $\tilde{\mathbf{w}}$ to minimize $\tilde{P}$ while ensuring the received SNR to be above a given target γ_o:

$$\begin{aligned} &\min_{\tilde{\mathbf{w}}} \quad \tilde{P} \\ &\text{subject to} \quad \frac{\tilde{P} \left| \tilde{\mathbf{w}}^H \mathbf{h} \right|^2}{\tilde{\mathbf{w}}^H \tilde{\mathbf{\Sigma}} \tilde{\mathbf{w}}} \geq \gamma_o. \end{aligned} \tag{16}$$

The duality between the multi-antenna relay beamforming and the direct point-to-point SIMO beamforming with uncertain noise and the same SNR requirement is established in the following.

Theorem 1: The Lagrange dual problem (14) associated with the optimization problem (9) is equivalent to the following problem:

$$\max_{\mathbf{\Lambda}} \min_{\nu, \tilde{\mathbf{w}}} \ \nu \sigma_d^2 \tag{17}$$

$$\text{subject to } \frac{\nu P_o \left| \tilde{\mathbf{w}}^H \mathbf{h} \right|^2}{\tilde{\mathbf{w}}^H \mathbf{\Sigma} \tilde{\mathbf{w}}} \geq \gamma_o \tag{18}$$

$$\mathrm{tr}\left(\mathbf{\Lambda}\right) \leq 1, \quad \mathbf{\Lambda} \text{ is diagonal} \tag{19}$$

$$\mathbf{\Lambda} \succ 0, \nu \geq 0 \tag{20}$$

where

$$\mathbf{\Sigma} \triangleq \mathbf{\Lambda} \otimes \left(P_o \mathbf{h}_1 \mathbf{h}_1^H + \sigma_r^2 \mathbf{I} \right) + \nu \left(\mathbf{h}_2 \mathbf{h}_2^H \otimes \sigma_r^2 \mathbf{I} \right). \tag{21}$$

Furthermore, the problem (17) can be interpreted as a point-to-point SIMO beamforming problem (16) with a dual transmit power $\tilde{P} = \nu \sigma_d^2$, the dual channel $\mathbf{h} = \mathbf{h}_2 \otimes \mathbf{h}_1$, and the noise covariance matrix $\tilde{\mathbf{\Sigma}} = \frac{\sigma_d^2}{P_o} \mathbf{\Sigma}$, for all diagonal $\mathbf{\Lambda} \succcurlyeq 0$ and $\mathrm{tr}\left(\mathbf{\Lambda}\right) \leq 1$, such that the SNR constraint (18) is satisfied.

Proof: The Lagrangian given in (13) can be rewritten as

$$\begin{aligned} &L(P_r, \mathbf{w}, \mathbf{\Lambda}, \nu) \\ &= \nu \sigma_d^2 + P_r[1 - \mathrm{tr}(\mathbf{\Lambda})] \\ &\quad + \mathbf{w}^H \left[\mathbf{\Lambda} \otimes (P_o \mathbf{h}_1 \mathbf{h}_1^H + \sigma_r^2 \mathbf{I}) - \nu \frac{P_o}{\gamma_o} \mathbf{h} \mathbf{h}^H + \nu \mathbf{R}_g \right] \mathbf{w} \\ &= \nu \sigma_d^2 + P_r[1 - \mathrm{tr}(\mathbf{\Lambda})] + \mathbf{w}^H \left[\mathbf{\Sigma} - \nu \frac{P_o}{\gamma_o} \mathbf{h} \mathbf{h}^H \right] \mathbf{w} \end{aligned} \tag{22}$$

where $\mathbf{\Sigma}$ is defined in (21). Substituting (22) into (14), we notice that the original dual problem (14) is equivalent to the following one with two new added constraints (19) and (24)

$$\max_{\mathbf{\Lambda}, \nu} \min_{P_r, \mathbf{w}} \ L(P_r, \mathbf{w}, \mathbf{\Lambda}, \nu) \tag{23}$$

$$\text{subject to } \text{(15) and (19)},$$

$$\mathbf{\Sigma} \succcurlyeq \frac{\nu P_o}{\gamma_o} \mathbf{h} \mathbf{h}^H. \tag{24}$$

This is because if either (19) or (24) is not satisfied, the inner minimization in (14) will result in $L(P_r, \mathbf{w}, \mathbf{\Lambda}, \nu) = -\infty$, which will not be the optimal solution of the dual problem (14). This implies that the optimal solution of (14) remains in the feasible set of the optimization problem (23). Thus, the problems (14) and (23) are equivalent. Following the above, it is straightforward to see that after the inner minimization of (23), the dual problem can now be expressed as

$$\max_{\mathbf{\Lambda}, \nu} \ \nu \sigma_d^2 \tag{25}$$

$$\text{subject to } \text{(15) (19) and (24)}.$$

To show (17) and (25) are equivalent, we first show the following equivalence.

Lemma 2: The dual problem (25) is equivalent to

$$\max_{\mathbf{\Lambda}} \max_{\nu} \ \nu \sigma_d^2 \tag{26}$$

$$\text{subject to } \text{(19) and (20)},$$

$$\frac{\nu P_o}{\gamma_o} \mathbf{h}^H \mathbf{\Sigma}^{\dagger} \mathbf{h} \leq 1. \tag{27}$$

Proof: See Appendix D.

To show (17) and (26) are equivalent, we adopt the general approach in [11]. We note that the inner minimization part in (17) can be interpreted as a direct point-to-point SIMO beamforming problem given in (16), where the transmit power is set as $\tilde{P} = \nu\sigma_d^2$, the channel is as $\mathbf{h} = \mathbf{h}_2 \otimes \mathbf{h}_1$, and the noise covariance matrix is as $\tilde{\mathbf{\Sigma}} = \frac{\sigma_d^2}{P_0}\mathbf{\Sigma}$. The solution of SIMO beamforming problem (16) is known, where the optimal receiver beamforming vector $\tilde{\mathbf{w}}^o$ is given by

$$\tilde{\mathbf{w}}^o = \tilde{\mathbf{\Sigma}}^\dagger \mathbf{h} = \frac{P_o}{\sigma_d^2}\mathbf{\Sigma}^\dagger \mathbf{h}. \tag{28}$$

Note that the optimal $\tilde{\mathbf{w}}^o$ is only unique up to a scale factor. That is, $\beta\tilde{\mathbf{w}}^o$ is also optimal for any arbitrary non-zero value of β. Substituting (28) into the SNR constraint (18), the optimization problem (17) is now expressed as

$$\max_{\mathbf{\Lambda}} \min_{\nu} \ \nu\sigma_d^2 \tag{29}$$

$$\text{subject to} \ \ \text{(19) and (20)},$$

$$\frac{\nu P_o}{\gamma_o}\mathbf{h}^H \mathbf{\Sigma}^\dagger \mathbf{h} \geq 1. \tag{30}$$

Comparing the problem (29) with (26), the SNR constraint is reversed and the maximization over ν is also reversed to minimization. With any fixed $\mathbf{\Lambda}$, we examine the inner minimization problem (29) and maximization problem (26). By substituting $\mathbf{\Sigma}$ with its definition in (21), we rewrite the SNR expression in (27) and (30) as

$$\begin{aligned}
&\frac{\nu P_o}{\gamma_o}\mathbf{h}^H \mathbf{\Sigma}^\dagger \mathbf{h} \\
&= \frac{\nu P_o}{\gamma_o}\mathbf{h}^H \left[\mathbf{\Lambda} \otimes \left(P_o \mathbf{h}_1 \mathbf{h}_1^H + \sigma_r^2 \mathbf{I}\right) + \nu\left(\mathbf{h}_2\mathbf{h}_2^H \otimes \sigma_r^2 \mathbf{I}\right)\right]^\dagger \mathbf{h} \\
&= \frac{P_o}{\gamma_o}\mathbf{h}^H \left[\frac{1}{\nu}\mathbf{\Lambda} \otimes \left(P_o \mathbf{h}_1 \mathbf{h}_1^H + \sigma_r^2 \mathbf{I}\right) + \left(\mathbf{h}_2\mathbf{h}_2^H \otimes \sigma_r^2 \mathbf{I}\right)\right]^\dagger \mathbf{h}
\end{aligned} \tag{31}$$

which is a monotonically increasing function of ν. This implies that the received SNR constraints in both problems (26) and (29) are met with equality at optimality, and the two problems lead to the same optimal ν^o which is the solution of

$$\frac{P_o}{\gamma_o}\mathbf{h}^H \left[\frac{1}{\nu}\mathbf{\Lambda} \otimes \left(P_o \mathbf{h}_1 \mathbf{h}_1^H + \sigma_r^2 \mathbf{I}\right) + \left(\mathbf{h}_2\mathbf{h}_2^H \otimes \sigma_r^2 \mathbf{I}\right)\right]^\dagger \mathbf{h} = 1. \tag{32}$$

This indicates that the two problems (29) and (26) are equivalent. By this, we have shown that the Lagrange dual problem (14) is equivalent to the problem (17). From Proposition 2, the optimal solution of the dual SIMO beamforming problem (17) is the same as that of the original relay beamforming problem (6). ∎

Let $P_{\max} \triangleq \max\{P_i\}$ in the problem (6). As mentioned in Section II-B, since the two problems (6) and (9) are equivalent, at optimality, the minimum powers in these two problems are equivalent, *i.e.*, $P_{\max}^o = P_r^o$. Following Proposition 2 and Theorem 1, we can obtain the value of the minimum per-antenna power at the relay through (17). The duality between multi-antenna relay beamforming and SIMO beamforming is shown in Fig. 2.

Corollary 1: The minimum per-antenna power P_r^o in the relay beamforming problem (9) is obtained through its dual point-to-point SIMO beamforming problem (17) as

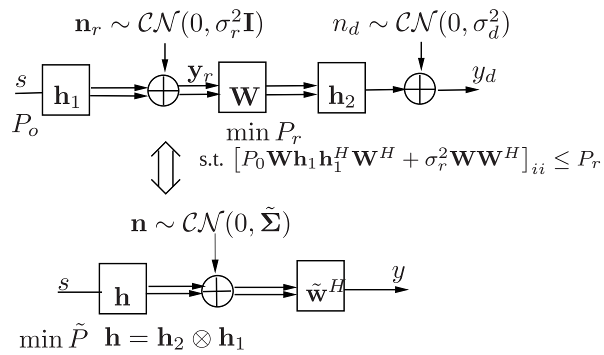


Fig. 2: Duality of relay beamforming and SIMO beamforming.

$$P_r^o = \nu^o \sigma_d^2 = \frac{\sigma_d^2 \gamma_o}{P_o \mathbf{h}^H \mathbf{\Sigma}^{o-1}\mathbf{h}} \tag{33}$$

where $\mathbf{\Sigma}^o$ is the value of $\mathbf{\Sigma}$ under the optimal $(\mathbf{\Lambda}^o, \nu^o)$.[7]

Proof: See Appendix E.

D. The Semi-Closed Form Solution for the Optimal $\mathbf{W}^o$

We have shown in Theorem 1 that the optimal relay processing matrix $\mathbf{W}^o$ under per-antenna power control in the problem (6) can be determined through its dual SIMO beamforming problem (17). The solution of the latter is given by (28), up to an arbitrary scale factor β. Thus, we obtain the optimal vectorized beamforming vector $\mathbf{w}^o$ for the problem (6) by

$$\mathbf{w}^o = \beta \mathbf{\Sigma}^{o-1}\mathbf{h} \tag{34}$$

under the optimal $(\mathbf{\Lambda}^o, \nu^o)$. To determine β, note that the SNR constraint (7) is met with equality at optimality. Using the SNR expression in (11), it follows that

$$\frac{P_o \mathbf{w}^{oH}\mathbf{h}\mathbf{h}^H\mathbf{w}^o}{\mathbf{w}^{oH}\mathbf{R}_\mathbf{g}\mathbf{w}^o + \sigma_d^2} = \gamma_o.$$

Substituting (34) into the above equation, we obtain

$$|\beta| = \sigma_d \left(\frac{P_o}{\gamma_o}\left(\mathbf{h}^H\mathbf{\Sigma}^{o-1}\mathbf{h}\right)^2 - \mathbf{h}^H\mathbf{\Sigma}^{o-1}\mathbf{R}_\mathbf{g}\mathbf{\Sigma}^{o-1}\mathbf{h}\right)^{-\frac{1}{2}}. \tag{35}$$

Since an arbitrary phase rotation in $\mathbf{w}$ due to β does not affect the SNR value, without loss of generality, we simply set $\beta = |\beta|$. Thus, we obtain the value of β as in (35).

By reversing the operation $\mathbf{w}^o = \mathrm{vec}(\mathbf{W}^{oH})$, we now have obtained the optimal relay processing matrix $\mathbf{W}^o$ of the relay power minimization problem (6). The solution has a closed-form expression once the optimal $(\mathbf{\Lambda}^o, \nu^o)$ are given. The determination of $(\mathbf{\Lambda}^o, \nu^o)$ however needs to be performed numerically as detailed in the next section.

Notice that the solution for β exists if and only if the denominator of the expression in (35) is valid. This in fact provides the necessary and sufficient condition for the existence of a

[7] From the proof of Lemma 2, $\mathbf{\Sigma}$ stratifying (20) is strictly positive definite.

feasible solution for the relay beamforming problem (6) which is shown in the following corollary.

Corollary 2: The necessary and sufficient condition for the multi-antenna relay beamforming problem (6) to be feasible is that there exists $\nu > 0$ and $\mathbf{\Lambda} \succ 0$ with $\mathrm{tr}(\mathbf{\Lambda}) \leq 0$, such that

$$\frac{P_o}{\gamma_o} \cdot \frac{\left(\mathbf{h}^H \mathbf{\Sigma}^{-1} \mathbf{h}\right)^2}{\mathbf{h}^H \mathbf{\Sigma}^{-1} \mathbf{R_g} \mathbf{\Sigma}^{-1} \mathbf{h}} > 1. \qquad (36)$$

Comparing (36) to (12), the condition requires to search for $(\mathbf{\Lambda}, \nu)$ and thus is difficult to be used to test the feasibility unless we solve the Lagrange dual problem given in (25). On the contrary, the condition (12) can be verified directly.

E. Determining $\mathbf{\Lambda}^o$ and ν^o through the Dual SDP

We have so far determined the structure of the optimal relay processing matrix $\mathbf{W}^o$ through $\mathbf{w}^o$ as given in (34). To determine the value of $\mathbf{W}^o$, we need to obtain the optimal $(\mathbf{\Lambda}^o, \nu^o)$. This can be done by directly solving the Lagrange dual problem (25). We show in the following proposition that the problem can be formulated and solved through semi-definite programming (SDP).

Proposition 3: The dual problem (25) is a dual SDP problem.

Proof: Define $\mathbf{s} \triangleq [0, \cdots, 0, -\sigma_d^2]^T$, $\mathbf{a} \triangleq [1, \cdots, 1, 0]^T$, where $\mathbf{s}, \mathbf{a} \in \mathbf{R}^{(N+1)\times 1}$, and $\mathbf{x} \triangleq [x_1, \cdots, x_N, x_{N+1}]^T = [\lambda_1, \cdots, \lambda_N, \nu]^T$. The constraint (24) can be expressed as

$$-\mathbf{\Lambda} \otimes \mathbf{R}_r - \nu \left(\mathbf{R}_g - \frac{P_0}{\gamma_0} \mathbf{h}\mathbf{h}^H \right) \preccurlyeq 0 \qquad (37)$$

where $\mathbf{R}_r \triangleq P_o \mathbf{h}_1 \mathbf{h}_1^H + \sigma_r^2 \mathbf{I}$. Observing that $\mathbf{\Lambda}$ is a diagonal matrix, we obtain

$$-\mathbf{\Lambda} \otimes \mathbf{R}_r = \sum_{i=1}^{N} \lambda_i \mathbf{F}_i,$$

where $\mathbf{F}_i \in \mathbf{C}^{N^2 \times N^2}$, for $i = 1, \cdots, N$, is a block diagonal matrix, whose ith diagonal block is given by $-\mathbf{R}_r$, and all other $(N-1)$ diagonal blocks are $N \times N$ zero matrices. Also define $\mathbf{F}_{N+1} \triangleq \frac{P_0}{\gamma_0} \mathbf{h}\mathbf{h}^H - \mathbf{R}_g$. Then, the constraint (37) can be further expressed as $\sum_{i=1}^{N+1} x_i \mathbf{F}_i \preccurlyeq 0$. Note that $\mathbf{F}_1, \cdots, \mathbf{F}_{N+1}$ are all Hermitian matrices. Therefore, the dual problem (25) can be transformed into the following SDP

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{s}^T \mathbf{x} && (38) \\ \text{subject to} \quad & \sum_{i=1}^{N+1} x_i \mathbf{F}_i \preccurlyeq 0 \\ & \mathbf{a}^T \mathbf{x} - 1 \leq 0, \quad \mathbf{x} \succcurlyeq 0. \end{aligned}$$

■

We have now converted the optimization problem (6) with N^2 variables and $(N+1)$ constraints to an SDP problem with $(N+1)$ variables. It can be solved efficiently using interior-point methods [12], for example, the logarithmic barrier method [25]. Standard SDP software such as SeDuMi [24] is based on interior-point methods and can be directly used to solve the problem. The complexity per iteration is $\mathcal{O}(N^6)$. As a comparison, for solving the primal problem (6) through SOCP in (55) directly using efficient interior-point methods, the complexity is $\mathcal{O}(N^7)$ per iteration[8]. Furthermore, for both SDP and SOCP, the number of iterations is known to be insensitive to problem size and typically lies between 5 and 50 [12], [26]. Thus, besides obtaining the insights on the solution structure, we see reduced complexity in finding the solution.

F. SNR Maximization

Instead of power minimization, we now consider the reverse problem of received SNR maximization with per-antenna power constraints $\{P_1, \cdots, P_N\}$, given by

$$\begin{aligned} \max_{\mathbf{W}} \quad & \mathrm{SNR} && (39) \\ \text{subject to} \quad & \left[P_o \mathbf{W}\mathbf{h}_1\mathbf{h}_1^H \mathbf{W}^H + \sigma_r^2 \mathbf{W}\mathbf{W}^H\right]_{ii} \leq P_i, && (40) \\ & \text{for } i = 1, \cdots, N \end{aligned}$$

where the expression of SNR is given in (4). The optimization problem (39) is always feasible. It can be solved through power minimization, based on the relation of the two problems.

Consider the following pair of reverse problems for SNR maximization and relay per-antenna power minimization

$$\begin{aligned} \max_{\mathbf{W}} \quad & \mathrm{SNR} && (41) \\ \text{subject to} \quad & \left[P_o \mathbf{W}\mathbf{h}_1\mathbf{h}_1^H \mathbf{W}^H + \sigma_r^2 \mathbf{W}\mathbf{W}^H\right]_{ii} \leq \eta P_i, && (42) \\ & \text{for } i = 1, \cdots, N. \end{aligned}$$

and

$$\begin{aligned} \min_{\mathbf{W}} \quad & \eta && (43) \\ \text{subject to} \quad & (42),\ \mathrm{SNR} \geq \gamma_o, \text{ and } 0 \leq \eta \leq 1. \end{aligned}$$

We now explicitly express the optimal objectives of (41) and (43) as functions of their corresponding constraint targets $\mathrm{SNR}^o(\eta)$ and $\eta^o(\gamma_o)$, respectively. The following shows the relation of the two problems.

Proposition 4: The SNR maximization problem (41) and the power minimization problem (43) under relay per-antenna power constraints are two inverse problems, i.e. $\eta^o(\mathrm{SNR}^o(\eta)) = \eta$ (or $\mathrm{SNR}^o(\eta^o(\gamma_o)) = \gamma_o$). Furthermore, the optimal objective $\mathrm{SNR}^o(\eta)$ (or $\eta^o(\gamma_o)$) is a continuous strictly monotonic increasing function of η.

Proof: See Appendix F. ■

By Proposition 4, the solution to the SNR maximization problem (41) can be obtained by iteratively solving the power minimization problem (9) with bisection search on the received SNR target γ_o. The stopping criterion for bisection search is when $\eta \to 1$. The procedure is summarized below.

1) Initialize $\gamma_{\min}$ and $\gamma_{\max}$. Set $\gamma_{\min}$ and $\gamma_{\max}$ such that the power minimization problem (43) is feasible and infeasible, respectively. Set ϵ.
2) Set $\gamma_o = (\gamma_{\max} + \gamma_{\min})/2$.

[8]Complexity analysis is based on standard SDP and SOCP problems. Depending on specific problems, special structure may be explored for improved efficiency.

3) If the problem (43) is infeasible under γ_o
   Set $\gamma_{\max} = \gamma_o$, $\eta = 0$ (or any value $\eta < 1 - \epsilon$).
   Else
   Set $\gamma_{\min} = \gamma_o$, $\eta = \eta^o(\gamma_o)$.
4) If $\eta < 1 - \epsilon$, repeat (2)-(4); otherwise, return γ_o.

G. Joint Design of $\mathbf{W}$, $\mathbf{b}$, and $\mathbf{r}$

The solution of $\mathbf{W}$ for relay beamforming developed so far assumes a given set of source and estimation beamforming vectors $(\mathbf{b}, \mathbf{r})$. For the joint optimization of $(\mathbf{W}, \mathbf{b}, \mathbf{r})$, finding a direct approach to solve it is challenging as the problem is nonconvex. Nonetheless, since the solution of the optimal $\mathbf{b}, \mathbf{r}$ with given $\mathbf{W}$ can be easily obtained, we can use the alternating optimization approach [27] for the joint design using the solution we obtained earlier:

1) For given $(\mathbf{b}, \mathbf{r})$, obtain the optimal $\mathbf{W}$ as in Section III-C-III-E;
2) Find optimal $\mathbf{b}$ and $\mathbf{r}$, based on $\mathbf{W}$ obtained in Step 1.
3) Repeat Step 1-2 until convergence.

Note that the above iteration is guaranteed to converge, although it may lead to a local maximum if there are multiple local maxima. Thus, such approach requires good initialization methods.

Note that even though joint optimization is desirable, it does not have to be performed at the relay to achieve an optimal system performance. For instance, the joint optimization can be done at the source and destination to obtain the optimal $(\mathbf{b}, \mathbf{r})$. The relay only needs to consider the optimization problem (6), using the equivalent channels at the first and second hops, *i.e.*, $\mathbf{h}_1$ and $\mathbf{h}_2$, respectively. In this case, $\mathbf{W}$ obtained at the relay would still be jointly optimal. The benefit of doing this, instead of letting the relay perform joint optimization, is two-fold: 1) Reduced computational complexity at the relay; 2) Reduced (feedback) overhead: the relay only needs to obtain the CSI of the equivalent channels $\mathbf{h}_1$ and $\mathbf{h}_2$, instead of $\mathbf{H}_1$ and $\mathbf{H}_2$.

IV. Relay Beamforming Achievable Rate

We now consider the maximum achievable rate under AF multi-antenna relaying with per-antenna power constraints, and its dual relation to that of a point-to-point system. The source-destination achievable rate R is directly related to the received SNR at the destination by $R = \frac{1}{2}\log(1 + \mathrm{SNR})$. Thus, the maximum achievable rate is obtained by finding the maximum received SNR at the destination under the per-antenna power constraints. We use the results obtained in Section III to establish the dual relation.

Consider the same system setup as described in Section II-A. To ease the explanation, we denote the AF MIMO relaying system as $\mathcal{S}^{\text{relay}}_{M_s \text{x} N \text{x} M_d}$, given by

$$\mathcal{S}^{\text{relay}}_{M_s \text{x} N \text{x} M_d} \begin{cases} \text{Source}: & M_s \text{ antennas, and per-antenna} \\ & \text{power constraint } P_o; \\ \text{Relay}: & N \text{ antennas, and per-antenna} \\ & \text{power constraint } P_r; \\ \text{Destination}: & M_d \text{ antennas;} \\ \text{Relay channels}: & \mathbf{H}_1, \mathbf{H}_2. \end{cases}$$

Note that the source antennas also have per-antenna power constraint, to be consistent with the per-antenna power constraint at the relay.

A. Single-Antenna Source and Destination

For source and destination each equipped with a single antenna, the channel vectors at the 1st and 2nd hops are $\mathbf{h}_1$ and $\mathbf{h}_2$, respectively, as in (2). The respective AF MIMO relaying system is given as $\mathcal{S}^{\text{relay}}_{1\text{x}N\text{x}1}$. We denote the dual SIMO system as $\mathcal{S}^{\text{SIMO}}_{1\text{x}N^2}$, given by

$$\mathcal{S}^{\text{SIMO}}_{1\text{x}N^2} \begin{cases} \text{Source}: & \text{single antenna, power constraint } P_r; \\ \text{Destination}: & N^2 \text{ antennas, and} \\ & \text{uncertain noise covariance } \frac{\sigma_d^2}{P_o}\mathbf{\Sigma} \\ \text{Channel}: & \mathbf{h}_2 \otimes \mathbf{h}_1 \end{cases}$$

where $\mathbf{\Sigma}$ is as in (21) with $\mathbf{\Lambda}$ and ν satisfying (19) and (20).

Combining Theorem 1 and Proposition 4, it is straightforward to see that the source-destination maximum achievable rate under multi-antenna relay beamforming with per-antenna constraints is the same as the maximum beamforming achievable rate of a corresponding dual SIMO channel.

Theorem 2: The maximum achievable rate of an AF multi-antenna relaying system $\mathcal{S}^{\text{relay}}_{1\text{x}N\text{x}1}$ is identical to the maximum achievable rate of a dual SIMO system $\mathcal{S}^{\text{SIMO}}_{1\text{x}N^2}$.

Note that the maximum achievable rate of a SIMO system is well known in the literature [28].

B. Single-Antenna Source with Multi-Antenna Destination

In this case, the destination has M_d receive antennas and uses a beamformer $\mathbf{r}$ for receive beamforming. The channels at the 1st and 2nd hops are $\mathbf{h}1$ and $\mathbf{H}_2$, respectively. The AF MIMO relaying system is given as $\mathcal{S}^{\text{relay}}_{1\text{x}N\text{x}M_d}$. We denote the dual MIMO system as $\mathcal{S}^{\text{MIMO}}_{M_d\text{x}N^2}$, given by

$$\mathcal{S}^{\text{MIMO}}_{M_d\text{x}N^2} \begin{cases} \text{Source}: & M_d \text{ antennas, total power constraint } P_r \\ \text{Destination}: & N^2 \text{ antennas, and} \\ & \text{uncertain noise covariance } \frac{\sigma_d^2}{P_o}\mathbf{\Sigma} \\ \text{Channel}: & \mathbf{H}_2^T \otimes \mathbf{h}_1 \end{cases}$$

where $\mathbf{\Sigma}$ is as in (21) with $\mathbf{\Lambda}$ and ν satisfying (19) and (20), and $\mathbf{h}_2$ in (21) is replaced by $(\mathbf{r}^H\mathbf{H}_2)^T$ for a given unitary transmit beamforming vector $\mathbf{r}$. Note that the maximum beamforming achievable rate of a MIMO system is well known in literature [28].

Theorem 3: The maximum achievable rate of an AF multi-antenna relaying system $\mathcal{S}^{\text{relay}}_{1\text{x}N\text{x}M_d}$ with receive beamforming is identical to the maximum beamforming achievable rate of a dual MIMO system $\mathcal{S}^{\text{MIMO}}_{M_d\text{x}N^2}$ with transmit and receive beamforming.

Proof: For a given beamforming vector $\mathbf{r}$ at the destination, the equivalent channel over the second hop is $\mathbf{h}_2 = (\mathbf{H}_2^H\mathbf{r})^*$, and the system is converted to an equivalent single-antenna destination system with channels $\mathbf{h}_1$, $\mathbf{h}_2$. Thus, by Theorem 2, the achievable rate of $\mathcal{S}^{\text{relay}}_{1\text{x}N\text{x}M_d}$, for any given $\mathbf{r}$, is the same as a dual SIMO system with channel $(\mathbf{r}^H\mathbf{H}_2)^T \otimes \mathbf{h}_1$. Since

$$(\mathbf{r}^H\mathbf{H}_2)^T \otimes \mathbf{h}_1 = \text{vec}(\mathbf{h}_1(\mathbf{r}^H\mathbf{H}_2)) = (\mathbf{H}_2^T \otimes \mathbf{h}_1)\mathbf{r}^* \quad (44)$$

where the second equation is by the fact that $\text{vec}(\mathbf{AXB}) = (\mathbf{B}^T \otimes \mathbf{A})\text{vec}(\mathbf{X})$. Thus, from (44), it is clear that the dual SIMO system with channel $(\mathbf{r}^H\mathbf{H}_2)^T \otimes \mathbf{h}_1$ is equivalent to a MIMO system with transmit beamforming, where the MIMO channel is given by $\mathbf{H}_2^T \otimes \mathbf{h}_1$ and the transmit beamforming vector is $\mathbf{r}^*$. Therefore, the achievable rate of $\mathcal{S}^{\text{relay}}_{1\text{x}N\text{x}M_d}$, for any given $\mathbf{r}$, is the same as the beamforming achievable rate of a dual MIMO system $\mathcal{S}^{\text{MIMO}}_{M_d\text{x}N^2}$, with transmit beamforming vector $\mathbf{r}^*$ (and the optimal receive beamforming).

The maximum beamforming achievable rate is obtained by maximizing the rate over all possible beamforming vector $\mathbf{r}$. Since $\mathcal{S}^{\text{relay}}_{1\text{x}N\text{x}M_d}$ and $\mathcal{S}^{\text{MIMO}}_{M_d\text{x}N^2}$ have the same achievable rate for any given $\mathbf{r}$, it is clear that they have identical maximum beamforming achievable rate. ∎

C. Multi-Antenna Source with Single-Antenna Destination

Similar to Section IV-B, the channels at the 1st and 2nd hops are $\mathbf{H}_1$ and $\mathbf{h}_2$, respectively. The AF MIMO relaying system is given as $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}1}$. Denote the dual MIMO system as $\mathcal{S}^{\text{MIMO}}_{M_s\text{x}N^2}$, given by

$$\mathcal{S}^{\text{MIMO}}_{M_s\text{x}N^2}\begin{cases}\text{Source}: & M_s \text{ antennas,}\\ & \text{per-antenna power constraint } \frac{P_r}{M_s}\\ \text{Destination}: & N^2 \text{ antennas, and}\\ & \text{uncertain noise covariance } \frac{\sigma_d^2}{P_o}\mathbf{\Sigma}\\ \text{Channel}: & \mathbf{h}_2 \otimes \mathbf{H}_1\end{cases}$$

where $\mathbf{\Sigma}$ is as in (21) with $\mathbf{\Lambda}$ and ν satisfying (19) and (20), and $\mathbf{h}_1$ in (21) is replaced by $\mathbf{H}_1\mathbf{b}$ for a given unitary transmit beamforming vector $\mathbf{b}$.

Theorem 4: The maximum achievable rate of an AF multi-antenna relaying system $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}1}$ with transmit beamforming is identical to the maximum beamforming achievable rate of a dual MIMO system $\mathcal{S}^{\text{MIMO}}_{M_s\text{x}N^2}$ with transmit and receive beamforming.

Proof: Similar to the proof of Theorem 3, the equivalent channel over the 1st hop is $\mathbf{h}_1 = \mathbf{H}_1\mathbf{b}$. For a given $\mathbf{b}$, the achievable rate of $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}1}$ is the same as a dual SIMO system with channel $\mathbf{h}_2 \otimes (\mathbf{H}_1\mathbf{b})$, and

$$\mathbf{h}_2 \otimes (\mathbf{H}_1\mathbf{b}) = (\mathbf{h}_2 \otimes \mathbf{H}_1)\mathbf{b}. \quad (45)$$

The RHS of the above equation can be viewed as a MIMO channel $\mathbf{h}_2 \otimes \mathbf{H}_1$ with transmit beamforming vector $\mathbf{b}$. Therefore, the achievable rate of $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}1}$ is equivalent to a dual MIMO system $\mathcal{S}^{\text{MIMO}}_{M_s\text{x}N^2}$, for any given $\mathbf{b}$. It follows that the maximum beamforming achievable rate, maximized over all possible $\mathbf{b}$, is the same for $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}1}$ and $\mathcal{S}^{\text{MIMO}}_{M_s\text{x}N^2}$. ∎

D. Multi-Antenna Source and Destination

Now assume the source and the destination have M_s and M_d antennas for transmit and receive beamforming, respectively. The MIMO relay channels at the two hops are $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. Let $\mathbf{b}$ and $\mathbf{r}$ be the transmit and receive beamforming vectors at the source and destination of $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}M_d}$, respectively. For given $\mathbf{b}$ and $\mathbf{r}$, $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}M_d}$ can be converted to an equivalent system $\mathcal{S}^{\text{relay}}_{1\text{x}N\text{x}1}$ with single-antenna source and destination, with equivalent channels at the first hop and second hop being $\widetilde{\mathbf{h}}_1 = \mathbf{H}_1\mathbf{b}$ and $\widetilde{\mathbf{h}}_2 = (\mathbf{r}^H\mathbf{H}_2)^T$. Thus, by Theorem 2, the maximum beamforming achievable rate of $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}M_d}$ is the same as a dual SIMO system with channel $(\mathbf{r}^H\mathbf{H}_2)^T \otimes (\mathbf{H}_1\mathbf{b})$. The dual SIMO channel can be rewritten as

$$\begin{aligned}(\mathbf{r}^H\mathbf{H}_2)^T \otimes (\mathbf{H}_1\mathbf{b}) &= \text{vec}((\mathbf{H}_1\mathbf{b})(\mathbf{r}^H\mathbf{H}_2))\\ &= (\mathbf{H}_2^T \otimes \mathbf{H}_1)\text{vec}(\mathbf{b}\mathbf{r}^H)\\ &= (\mathbf{H}_2^T \otimes \mathbf{H}_1)(\mathbf{r}^* \otimes \mathbf{b}). \quad (46)\end{aligned}$$

The expression in (46) indicates that we can view the dual system as a dual MIMO channel with transmit beamforming, where the dual MIMO channel is $\mathbf{H}_2^T \otimes \mathbf{H}_1$ and the transmit beamforming vector is $\mathbf{r}^* \otimes \mathbf{b}$. The dual MIMO system, denoted as $\mathcal{S}^{\text{MIMO}}_{M_sM_d\text{x}N^2}$, is described by

$$\mathcal{S}^{\text{MIMO}}_{M_sM_d\text{x}N^2}\begin{cases}\text{Source}: & M_sM_d \text{ antennas,}\\ & \text{total power constraint } P_r\\ \text{Destination}: & N^2 \text{ antennas, and}\\ & \text{uncertain noise covariance } \frac{\sigma_d^2}{P_o}\mathbf{\Sigma}\\ \text{Channel}: & \mathbf{H}_2^T \otimes \mathbf{H}_1\end{cases}$$

where the covariance matrix $\mathbf{\Sigma}$ is as in (21) with $\mathbf{\Lambda}$ and ν satisfying (19) and (20), and $\mathbf{h}_1$ and $\mathbf{h}_2$ in (21) are replaced by $\mathbf{H}_1\mathbf{b}$ and $(\mathbf{H}_2^H\mathbf{r})^*$ for given beamforming vectors $\mathbf{b}$ and $\mathbf{r}$, respectively. Thus, the achievable rate of $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}M_d}$, for any given $\mathbf{b}$ and $\mathbf{r}$, is the same as the maximum beamforming achievable rate of a dual MIMO system $\mathcal{S}^{\text{MIMO}}_{M_sM_d\text{x}N^2}$ with transmit beamforming vector being $\mathbf{r}^* \otimes \mathbf{b}$.

Notice that the equivalence of the dual MIMO system $\mathcal{S}^{\text{MIMO}}_{M_sM_d\text{x}N^2}$ and the MIMO relay system $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}M_d}$, as established above, limits the transmit beamforming vector to the form $\mathbf{r}^* \otimes \mathbf{b}$. Clearly, by choosing all possible $(\mathbf{b}, \mathbf{r})$, the resulting beamforming vector set is strictly a subset of all possible MIMO beamforming vectors $\mathbf{b}_{\text{MIMO}}$'s, *i.e.*, $\{\mathbf{r}^* \otimes \mathbf{b}\} \subset \{\mathbf{b}_{\text{MIMO}}\}$. Thus, instead of establishing a direct equivalence between the maximum beamforming achievable rate of $\mathcal{S}^{\text{relay}}_{M_s\text{x}N\text{x}M_d}$ and that of $\mathcal{S}^{\text{MIMO}}_{M_sM_d\text{x}N^2}$, we are only able to show the following bound.

Proposition 5: The maximum beamforming achievable rate of an AF MIMO relaying system $\mathcal{S}^{\text{relay}}_{M_s\times N\times M_d}$ with transmit and receive beamforming at source and destination is upper bounded by the maximum beamforming achievable rate of a dual MIMO system $\mathcal{S}^{\text{MIMO}}_{M_sM_d\times N^2}$ with transmit and receive beamforming.

V. Numerical Results

A. Power Usage under Per-Antenna or Sum-Power Budget

We compare the performance under per-antenna power minimization of P_r in (9) with that under the sum-power P_{sum} minimization at the relay, where $P_{\text{sum}} = NP_r$. The source and destination are equipped with single antenna. Although the problem of sum-power minimization (or its reverse problem of SNR maximization) has a clear closed-form solution, most practical applications are constrained by the per-antenna power budgets. Our purpose is to analyze the difference of the power usage statistics under the two cases.

The solution of relay processing design for destination SNR maximization under a relay sum-power constraint is obtained in [4]. The optimal relay processing matrix, denoted as $\mathbf{W}^o_{\rm sum}$, has a closed-form solution. Under $\mathbf{W}^o_{\rm sum}$, the maximum destination SNR is obtained. Using this result, the corresponding minimum sum-power $P_{\rm sum}$ for a given destination SNR target γ_o is readily obtained as $P_{\rm sum} = \frac{\gamma_o\sigma_d^2(\sigma_r^2+P_o\|\mathbf{h}_1\|^2)}{(P_o\|\mathbf{h}_1\|^2-\sigma_r^2\gamma_o)\|\mathbf{h}_2\|^2}$.

In our simulations, the noise powers at the relay and at the destination are set to be equal $\sigma_r^2 = \sigma_d^2$, and we set the source transmitted power and noise variance such that $P_o/\sigma_r^2 = 10$dB. The entries of $\mathbf{h}_1$ and $\mathbf{h}_2$ are assumed i.i.d. complex Gaussian with zero-mean and variance 1.

Fig. 3 shows the average per-antenna power usage vs. required SNR target γ_0 for the number of relay antennas $N = 2, 4, 6$ under both power objectives. The average per-antenna power usage is averaged over all antennas over 10^4 channel realizations. As shown in the plot, the sum-power objective results in less average power usage than the per-antenna power objective does due to the flexibility of power distribution among antennas. The gap increases as the number of antennas N increases. Nonetheless, the small gaps indicate that the power usages in the mean sense are close under the two type of constraints.

To study the statistical behavior of antenna power usage under both types of power minimization, we presents in Fig. 4 the probability density function (PDF) of power usage on the first antenna at the relay, for $N = 2, 4, 6$. As we see the variance of power usage on a fixed antenna under the per-antenna power control case is much smaller than that under the sum-power control case. Heavier left tails for the sum-power case indicates on average more power usage on each fixed antenna when per-antenna power control is imposed. Overall, the per-antenna objective results in less peak-to-average power consumption per antenna than the sum-power objectees does. The difference increases with the number of antennas at the relay.

In Fig. 5, we compare the PDF of the maximum power usage among all antennas at the relay under both types of power minimization objectives. We see a clear shift of power profile of the maximum power consumption among antennas under the two cases, where the case with per-antenna power control results in a lower peak power consumption. This indicates per-antenna power control results in better balance of the power usage among antennas than the sum-power control does.

B. Centralized vs. Distributed Relay Beamforming

With the solution of the optimal relay processing matrix under relay per-antenna power budget obtained, we can now compare the performance of a centralized relay beamforming system with that of a distributed relay beamforming system to quantify the loss due to the distributed nature of processing. Such comparison is unavailable in previous studies that only consider a sum power constraint.

For the source and destination equipped with single-antenna, consider the centralized system with a single N-antenna relay[9], and the distributed system with N single-antenna relays. Per-antenna power budget is considered in both systems. For the centralized case, the advantage lies in the joint processing of received signals at the relay, while in the distributed case, each relay can only process its own received signals. We are interested to quantify the loss of such distributed nature of relay beamforming as compared to the centralized joint processing. The solution for the optimal distributed relay beamforming under individual relay power budget for SNR maximization was obtained analytically in [29] (an alternative numerical approach is given in [30]). Using this result, we compare the performance of the two systems.

Let $\text{SNR}_{\rm centr}$ and $\text{SNR}_{\rm distr}$ denote the maximum SNR achieved under the centralized and distributed relay beamforming systems, respectively. Fig.6(a) shows $\mathrm{E}[\text{SNR}_{\rm distr}/\text{SNR}_{\rm centr}]$ vs. P_o/σ_r^2, the average received SNR at each relay antenna for $N = 2, 4, 8$; Fig. 6(b) shows the corresponding $E[\text{SNR}_{\rm distr}]$ and $E[\text{SNR}_{\rm centr}]$ vs. P_o/σ_r^2. We vary the received SNR at relay by changing noise variance σ_r^2 while fixing the transmit power at the source $P_o = 10$dBW. It can be seen that the loss due to distributed processing first increases from the noiseless (high SNR) to noisy case, then decreases as the noise becomes high (low SNR). Intuitively, joint processing helps reduce effective noise when noise level is moderate, but it becomes ineffective when the noise becomes dominant. The biggest loss is approximately 1dB-2dB for $N = 2$ to 8. It is particular to note that the biggest loss happens in the range of SNR which is typical in the practical systems.

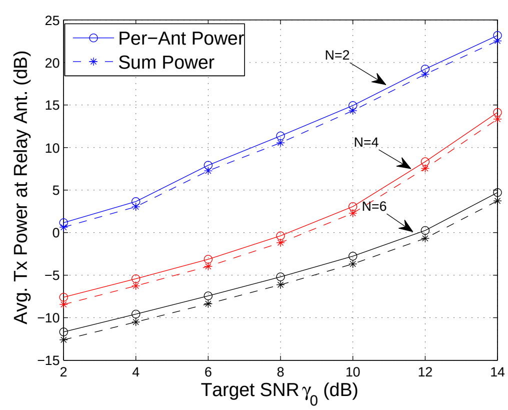


Fig. 3: Average per-antenna power usage vs. required SNR γ_0.

VI. Conclusion

In this paper, we have investigated the design of multi-antenna relay processing matrix for unicast AF MIMO relay

[9]We can also consider it as a system with N single-antenna relays with joint processing capability.

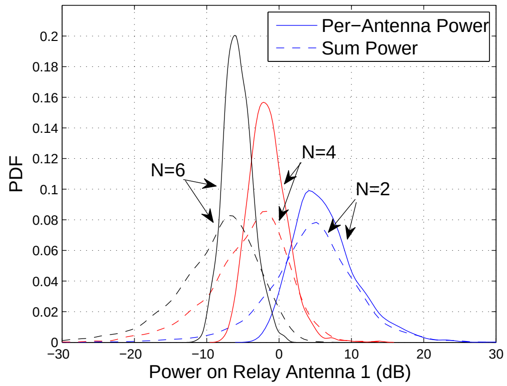


Fig. 4: PDF of power usage on the first antenna ($\gamma_o = 10$dB).

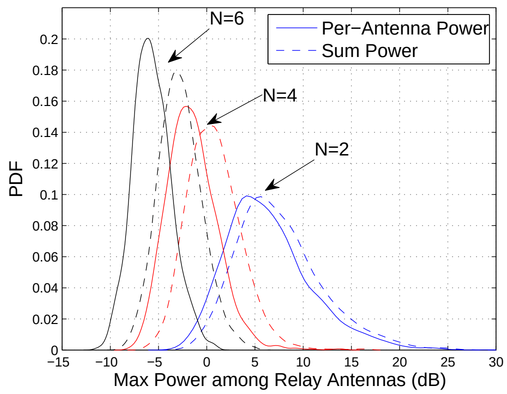
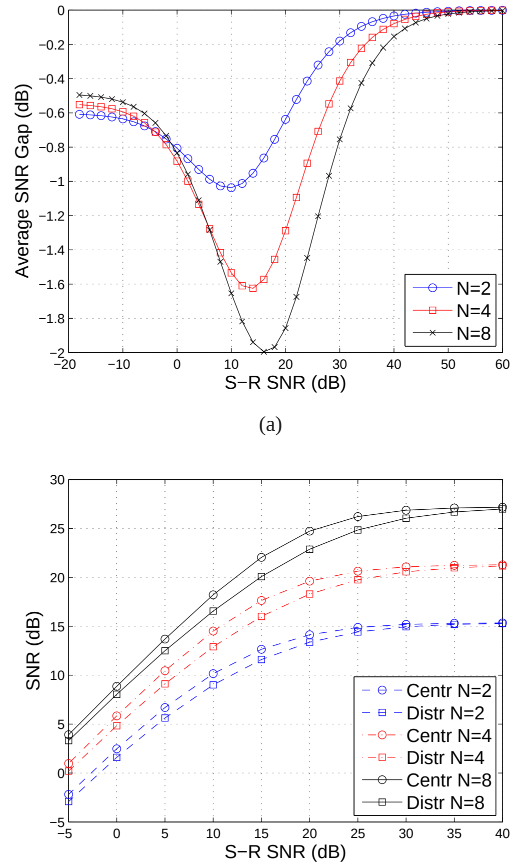


Fig. 5: PDF of maximum power usage among antennas ($\gamma_o = 10$dB).



(a)

(b)

Fig. 6: Performance gap between centralized vs. distributed relay Beamforming: (a) Average received SNR gap vs. 1st hop received SNR; (b) Average SNR vs. 1st hop received SNR.

beamforming with per-antenna power control. By transforming the original non-convex problem and solving it in the Lagrange dual domain, we have established the duality between relay beamforming system with (equivalent) single-antenna source and destination and direct SIMO beamforming system with uncertain noise covariance. This enables us to obtain a semi-closed form solution of the optimal relay processing matrix parameterized by the Lagrange dual variables. The solution not only reveals the structure of the optimal processing matrix, but also allows us to reduce the computational complexity of the original problem, through drastic reduction in the number of optimization variables and constraints, as well as an efficient SDP formulation of the dual problem to determine the dual variables. Following this, both SNR maximization problem with given per-antenna power budgets and joint optimization of relay processing matrix and source/destination beamforming vectors for MIMO relay beamforming are discussed. We have then examined the beamforming achievable rate in general MIMO relaying systems with multi-antenna source/destination. The duality relation of the maximum beamforming achievable rate of the MIMO relaying system and that of the direct MIMO system is established for scenarios with different antenna setups at the source and the destination.

Appendix A
Proof of Lemma 1

Proof: Following the property $(\mathbf{A} \otimes \mathbf{B}^T)\text{vec}(\mathbf{X}^T) = \text{vec}(\mathbf{AXB})$ for matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{X}$, we have

$$\mathbf{h}_2^T\mathbf{W}\mathbf{h}_1 = (\mathbf{h}_2 \otimes \mathbf{h}_1)^T\text{vec}(\mathbf{W}^T) = \left(\mathbf{h}^H\mathbf{w}\right)^* \tag{47}$$

where $\mathbf{w} = \text{vec}(\mathbf{W}^H)$. Similarly,

$$\mathbf{h}_2^T\mathbf{W} = \mathbf{h}_2^T\mathbf{W}\mathbf{I} = (\mathbf{h}_2 \otimes \mathbf{I})^T\text{vec}(\mathbf{W}^T) = \left((\mathbf{h}_2 \otimes \mathbf{I})^H\mathbf{w}\right)^*.$$

Therefore, we have

$$\sigma_r^2\|\mathbf{h}_2^T\mathbf{W}\|^2 = \mathbf{w}^H(\mathbf{h}_2\mathbf{h}_2^H \otimes \mathbf{I}\sigma_r^2)\mathbf{w} = \mathbf{w}^H\mathbf{R}_\mathbf{g}\mathbf{w} \tag{48}$$

where we have used the property $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ to arrive at the second equation. Substituting (47) and (48) in (4), we have (11). ■

Appendix B
Proof of Proposition 1

Proof: Note that $\mathbf{R}_\mathbf{g}$ is a positive semi-definite matrix. A feasible solution for $\mathbf{w}$ will not be in the null space of $\mathbf{R}_\mathbf{g}$, denoted as null$\{\mathbf{R}_\mathbf{g}\}$. If $\mathbf{w} \in$ null$\{\mathbf{R}_\mathbf{g}\}$, *i.e.*, $\mathbf{R}_\mathbf{g}^{\frac{1}{2}}\mathbf{w} = \mathbf{0}$, from (4) it is clear that $\mathrm{SNR} = 0$. Thus, we only consider $\mathbf{w} \notin \text{null}\{\mathbf{R_g}\}$.

For $\mathbf{w} \notin \text{null}\{\mathbf{R_g}\}$, consider the following upper bound for the received SNR as in (11) of Lemma 1

$$\mathrm{SNR_{up}} = \frac{P_o|\mathbf{h}^H\mathbf{w}|^2}{\left\|\mathbf{R}_\mathbf{g}^{\frac{1}{2}}\mathbf{w}\right\|^2}. \tag{49}$$

The optimization problem (6) is feasible only if there exists $\mathbf{w}$ (or $\mathbf{W}$), satisfying per-antenna power constraint (8), such that $\mathrm{SNR_{up}} > \gamma_o$. The per-antenna power constraint (10) can always be satisfied by scaling $\mathbf{w}$ as $c\mathbf{w}$ (can be seen from (53)), without changing the value of $\mathrm{SNR_{up}}$ in (49). This means the problem (6) is feasible only if there exists $\mathbf{w}$, such that

$$\max_{\mathbf{w}\notin\text{null}\{\mathbf{R_g}\}} \frac{P_o\mathbf{w}^H\mathbf{h}\mathbf{h}^H\mathbf{w}}{\mathbf{w}^H\mathbf{R_g}\mathbf{w}} > \gamma_o. \tag{50}$$

The LHS of (50) is a generalized eigenvalue problem with $\mathbf{R_g}$ in singular form [31]. The objective function is maximized when $\mathbf{w}$ is the principle generalized eigenvector $\mathbf{R}_\mathbf{g}^\dagger\mathbf{h}$, and the maximum value of LHS is given by $P_o\mathbf{h}^H\mathbf{R}_\mathbf{g}^\dagger\mathbf{h}$. Thus, we need

$$\frac{P_o}{\gamma_o}\mathbf{h}^H\mathbf{R}_\mathbf{g}^\dagger\mathbf{h} > 1. \tag{51}$$

Observing the structure of the LHS, we have

$$\begin{aligned}\mathbf{h}^H\mathbf{R}_\mathbf{g}^\dagger\mathbf{h} &= (\mathbf{h}_2^H \otimes \mathbf{h}_1^H)\left(\left(\mathbf{h}_2\mathbf{h}_2^H\right)^\dagger \otimes \frac{1}{\sigma_r^2}\mathbf{I}\right)(\mathbf{h}_2 \otimes \mathbf{h}_1) \\ &= \left(\mathbf{h}_2^H\left(\mathbf{h}_2\mathbf{h}_2^H\right)^\dagger\mathbf{h}_2\right) \otimes \left(\frac{1}{\sigma_r^2}\mathbf{h}_1^H\mathbf{h}_1\right) \\ &= \frac{\mathbf{h}_1^H\mathbf{h}_1}{\sigma_r^2}\end{aligned} \tag{52}$$

where the first equation follows the fact that $(\mathbf{A}\otimes\mathbf{B})^\dagger = \mathbf{A}^\dagger \otimes \mathbf{B}^{-1}$, for $\mathbf{A}$ being singular and $\mathbf{B}$ being invertible, and the second equation is from the fact $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC})\otimes(\mathbf{BD})$. Combining (51) and (52), we have (12). ■

APPENDIX C
PROOF OF PROPOSITION 2

Proof: We first show that the constraint function in (10) is convex. To see this, since $\mathbf{W}^H = [\mathbf{w}_1, \cdots \mathbf{w}_N]$, (10) can be rewritten as

$$\left[P_0\mathbf{W}\mathbf{h}_1\mathbf{h}_1^H\mathbf{W}^H + \sigma_r^2\mathbf{W}\mathbf{W}^H\right]_{ii} = \mathbf{w}_i^H(P_0\mathbf{h}_1\mathbf{h}_1^H + \sigma_r^2\mathbf{I})\mathbf{w}_i \tag{53}$$

for $i = 1, \cdots, N$. This shows that the constraint function is convex w.r.t. $\mathbf{w}_i$.

Note that $\mathbf{w} = \mathrm{vec}(\mathbf{W}^H) = [\mathbf{w}_1^H, \cdots, \mathbf{w}_N^H]^H$. From the SNR expression in (11), the constraint (7) is a non-convex function w.r.t. $\mathbf{w}$. However, it can be converted into an SOCP constraint [25]. To see this, the inequality in (7) can be rewritten as

$$P_o|\mathbf{h}^H\mathbf{w}|^2 \geq \gamma_o(\|\mathbf{R}_g^{\frac{1}{2}}\mathbf{w}\|^2 + \sigma_d^2) = \gamma_o\left\|\begin{bmatrix}\mathbf{R}_g^{\frac{1}{2}}\mathbf{w}\\ \sigma_d\end{bmatrix}\right\|^2.$$

Thus, we have

$$\sqrt{P_o}|\mathbf{w}^H\mathbf{h}| \geq \sqrt{\gamma_o}\left\|\begin{bmatrix}\mathbf{R}_g^{\frac{1}{2}}\mathbf{w}\\ \sigma_d\end{bmatrix}\right\|. \tag{54}$$

Since $\mathbf{w}$ is only unique up to a phase rotation, we can remove $|\cdot|$ from the left hand side (LHS) of (54), and assume $\mathbf{w}^H\mathbf{h}$ to be real. In this case, (54) is a SOCP constraint, and we have converted the optimization problem (9) to an SOCP problem

$$\min_{\mathbf{w}} P_r \tag{55}$$

$$\begin{aligned}&\text{subject to } (54) \\ &\qquad \mathbf{w}_i^H(P_0\mathbf{h}_1\mathbf{h}_1^H + \sigma_r^2\mathbf{I})\mathbf{w}_i \leq P_r, \text{ for } i = 1, \cdots, N.\end{aligned}$$

Note that strong duality holds for the conic form of an SOCP problem[10]. However, the constraint (54) is not in conic form, and the problem (55) is non-convex. In [32, Proposition 3], the optimality conditions on non-convex optimization problems with constraints in the form of (54) are given. The result there further implies that strong duality also holds for the SOCP problem (55) to its Lagrangian dual problem. To prove Proposition 2, we only need to show that the Lagrange dual of the problem (55) is the same as the Lagrange dual of the problem (9). We use a similar argument as in [11, Proposition 1] to show this.

The Lagrangian of (55) is

$$\begin{aligned}L'(P_r, \mathbf{w}, \lambda_i', \nu') = P_r + \nu'\left\{\left\|\begin{bmatrix}\mathbf{R}_g^{\frac{1}{2}}\mathbf{w}\\ \sigma_d\end{bmatrix}\right\| - \sqrt{\frac{P_o}{\gamma_o}}\left|\mathbf{w}^H\mathbf{h}\right|\right\} \\ + \sum_{i=1}^N \lambda_i'\{\mathbf{w}_i^H\left(P_o\mathbf{h}_1\mathbf{h}_1^H + \sigma_r^2\mathbf{I}\right)\mathbf{w}_i - P_r\}.\end{aligned} \tag{56}$$

The Lagrangian of the original problem (9) is given in (13). Comparing the two Lagrangians, the difference lies in the second term. Let

$$c \triangleq \left\|\begin{bmatrix}\mathbf{R}_g^{\frac{1}{2}}\mathbf{w}\\ \sigma_d\end{bmatrix}\right\| + \sqrt{\frac{P_o}{\gamma_o}}\left|\mathbf{w}^H\mathbf{h}\right| \geq \sigma_d.$$

We convert the second term in (56) as

$$\begin{aligned}&\nu'\left\{\left\|\begin{bmatrix}\mathbf{R}_g^{\frac{1}{2}}\mathbf{w}\\ \sigma_d\end{bmatrix}\right\| - \sqrt{\frac{P_o}{\gamma_o}}\left|\mathbf{w}^H\mathbf{h}\right|\right\} \\ &= \frac{\nu'}{c}\left\{\sigma_d^2 + \|\mathbf{R}_g^{\frac{1}{2}}\mathbf{w}\|^2 - \frac{P_o}{\gamma_o}\left|\mathbf{w}^H\mathbf{h}\right|^2\right\},\end{aligned} \tag{57}$$

and (56) becomes

$$\begin{aligned}L'(P_r, \mathbf{w}, \lambda_i', \nu') = P_r + \frac{\nu'}{c}\left\{\sigma_d^2 + \|\mathbf{R}_g^{\frac{1}{2}}\mathbf{w}\|^2 - \frac{P_o}{\gamma_o}\left|\mathbf{w}^H\mathbf{h}\right|^2\right\} \\ + \sum_{i=1}^N \lambda_i'\{\mathbf{w}_i^H\left(P_o\mathbf{h}_1\mathbf{h}_1^H + \sigma_r^2\mathbf{I}\right)\mathbf{w}_i - P_r\}.\end{aligned}$$

Since c is lower bounded by $\sigma_d > 0$, for any $\nu' \in [0, \infty)$, by changing the variable $\nu = \nu'/c$, there exists $\nu \in [0, \infty)$, such that we arrive at the expression exactly the same as the Lagrangian for (9) (also shown in (13)). Thus, the optimization problem (9) has zero duality gap to its dual problem. ■

[10] The conic form of an SOCP problem (*i.e.*, with conic inequalities) is convex [25], and strong duality holds between the conic-form primal problem and its conic dual problem

Appendix D
Proof of Lemma 2

Proof: Using (21) and the definition of $\mathbf{h}$, the matrix inequality constraint (24) can be rewritten as

$$\mathbf{\Lambda} \otimes \left(P_o \mathbf{h}_1 \mathbf{h}_1^H + \sigma_r^2 \mathbf{I}\right) + \nu \mathbf{h}_2 \mathbf{h}_2^H \otimes \left(\sigma_r^2 \mathbf{I} - \frac{P_o}{\gamma_o} \mathbf{h}_1 \mathbf{h}_1^H\right) \succcurlyeq 0.$$

By the feasibility condition (12) in Proposition 1, $\left(\sigma_r^2 \mathbf{I} - \frac{P_o}{\gamma_o} \mathbf{h}_1 \mathbf{h}_1^H\right)$ is an indefinite matrix. Since all forwarding links are active, *i.e.*, $|h_{2i}| > 0$, $\forall i$, for the above condition to hold, we have $\lambda_i > 0$, $\forall i$, or $\mathbf{\Lambda} \succ 0$. Thus, any feasible solution of (25) need to satisfy (20), and we can replace (15) by (20).

From the expression of $\mathbf{\Sigma}$ in (21), the constraint (20) means $\mathbf{\Sigma}$ is positive definite. [11, Lemma 1] states that, for positive definite matrix $\mathbf{A}$, $\mathbf{A} \succcurlyeq \mathbf{b}\mathbf{b}^H \Leftrightarrow \mathbf{b}^H \mathbf{A}^{-1} \mathbf{b} \leq 1$[11]. Thus, the SNR constraint (24) under (20) is equivalent to

$$\frac{\nu P_0}{\gamma_0} \mathbf{h}^H \mathbf{\Sigma}^{-1} \mathbf{h} \leq 1. \tag{58}$$

The constraint (27) under (20) is also equivalent to (58). Consequently, the two optimization problems (25) and (26) are equivalent. ■

Appendix E
Proof of Corollary 1

Proof: At optimality, the minimum per-antenna power P_r^o in (9) is the same as the value of the objective function in (17). As we have shown in the proof of Theorem 1, the problem (17) is equivalent to the problem (26), and the solution of the latter is given by the solution of (32) or equivalently

$$\frac{\nu P_o}{\gamma_o} \mathbf{h}^H \mathbf{\Sigma}^{-1} \mathbf{h} = 1 \tag{59}$$

under the optimal $\mathbf{\Lambda}^o$. It follows that we have (33). ■

Appendix F
Proof of Proposition 4

Proof: We first consider the SNR maximization with a common per-antenna power constraint P_r

$$\max_{\mathbf{W}} \quad \text{SNR} \tag{60}$$

$$\text{subject to} \quad \left[P_o \mathbf{W} \mathbf{h}_1 \mathbf{h}_1^H \mathbf{W}^H + \sigma_r^2 \mathbf{W}\mathbf{W}^H\right]_{ii} \leq P_r, \quad \text{for } i = 1, \cdots, N. \tag{61}$$

It is a reverse problem of the power minimization problem (9). We explicitly express the optimal objectives of (9) and (60) as functions of their corresponding constraint targets $\text{SNR}^o(P_r)$ and $P_r^o(\gamma_o)$, respectively.

To show $\text{SNR}^o(P_r)$ is strictly monotonically increasing with P_r, we prove it by contradiction. Assume $\tilde{P}_r > P_r$ and $\text{SNR}^o(\tilde{P}_r) \leq \text{SNR}^o(P_r)$, for some $\tilde{P}_r$ and P_r. Let $\mathbf{W}^o$ be the optimal processing matrix achieving $\text{SNR}^o(P_r)$. We can multiply $\mathbf{W}^o$ by a scalar $0 < c < 1$ so that the resulting SNR equals $\text{SNR}^o(\tilde{P}_r)$; at the same time, the per-antenna power usage in (10) under $c\mathbf{W}^o$ is $c^2 P_r < \tilde{P}_r$, because $P_r < \tilde{P}_r$. This contradicts the assumption that $\tilde{P}_r$ is optimal for $\gamma = \text{SNR}^o(\tilde{P}_r)$. It is also straightforward to show $\text{SNR}^o(P_r)$ is continuous w.r.t. P_r. Furthermore, we show that any $\gamma \leq \text{SNR}^o(P_r)$ is achievable. By scaling $\mathbf{W}^o$ with $c > 0$, we obtain $\mathbf{W} = c\mathbf{W}^o$. Let c be

$$c = \frac{\sigma_d^2}{\frac{P_o}{\gamma} |\mathbf{h}_2^T \mathbf{W}^o \mathbf{h}_1|^2 - \sigma_r^2 \|\mathbf{h}_2^T \mathbf{W}^o\|^2} > 0$$

where the denominator is positive since $\gamma < \text{SNR}^o(P_r)$. It can be verified that the corresponding received SNR is γ.

Since the optimal objective $\text{SNR}^o(P_r)$ is continuous and strictly monotonically increasing with P_r, and any $\gamma < \text{SNR}^o(P_r)$ is achievable, it is clear that for any given γ_o, the minimum per-antenna power P_r is attained when $\text{SNR}^o(P_r) = \gamma_0$, i.e. $P_r^o(\text{SNR}^o(P_r)) = P_r$. Thus, the problems (9) and (60) are inverse problems.

For the SNR maximization problem (41), the per-antenna power constraint (42) can be rewritten as

$$\frac{1}{P_i} \left[P_o \mathbf{W} \mathbf{h}_1 \mathbf{h}_1^H \mathbf{W}^H + \sigma_r^2 \mathbf{W}\mathbf{W}^H\right]_{ii} \leq \eta \tag{62}$$

which has the same form as the constraint (61). Furthermore, the optimization problem (43) is essentially in the same form of the optimization problem (9), with an additional condition of $\eta \in [0, 1]$. Thus, we can directly apply the above result for the common per-antenna constraint case to the pair of optimization problems (41) and (43). ■

[11] Note that the original [11, Lemma 1] only assumes $\mathbf{A}$ to be positive semidefinite. However, in fact, the equivalence only holds for positive definite $\mathbf{A}$ in general.

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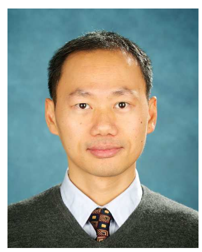

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