Gaussian: Generally Assumed, Ubiquitous, Symmetric Shape Identifying Additive Noise-Dr. O. Lord
Question: Why did you use the Gaussian assumption?
Answer: Because it's the normal assumption!-Overheard at a conference

THE BELL-SHAPED CURVE, USUALLY LABELED as normal or Gaussian, occupies a unique place in mathematics as well as in electrical engineering. The special properties that the shape possesses and the many optimal criteria it satisfies elevate it to this honored position. Let us sample some of the marvels of normalcy and sing the glories of Gaussianity!

The credit for discovering the Gaussian curve should actually go to Abraham De Moivre (1667-1754). (Another discovery of De Moivre, but again not credited to him, is the Stirling formula for approximating factorials of large numbers.) De Moivre' s *Annuities upon Lives* and *Doctrine of Chances* were pioneering works in the fields of actuarial mathematics and probability theory. Since De Moivre played such an important role in the history of mathematics when he was alive, it is but fitting that mathematics should play a role in his death. According to popular belief, De Moivre noticed that he was sleeping every day 15 minutes more than on the previous day. When the arithmetic progression reached 24 hours, De Moivre slipped into eternal sleep [1].

De Moivre' s discovery of the normal distribution in 1733 went unnoticed, and Gauss and Laplace rediscovered it in 1809 and 1812, respectively, in connection with their work on the theory of errors of observation. Their formulation of the law of errors is, in essence, the same at the presently well-known central limit theorem in probability. The theorem states that, under very general conditions, the arithmetic mean of a large number of independent random variables is approximately normally distributed. The importance of the normal distribution is underscored by a famous remark by Lippman [2]: "Everybody believes in the law of errors, the experimenters because they think it is a mathematical theorem, the mathematicians because they think it is an experimental fact."

Besides its undoubtedly central role in probability, the normal shape itself exhibits many important attributes-even in a deterministic setting. Probably the most fascinating property is the shape-invariance under Fourier transformation. In other words, the Gaussian pulse is an eigenfunction of the Fourier transform operation. Incidentally, there are not too many functions that share such a property. Hermite functions (the first member of the Hermite family is the Gaussian curve) and prolate spheroidal wave functions are some others that have the same shape in both time and frequency domains. It is intuitive that such invariance must be tied to some optimality criteria. For example, it is known that the Gaussian pulse possesses the smallest time duration-bandwidth product, with time and frequency durations defined in a mean square sense [3]. Simultaneous concentration in both domains is a desirable property in many applications and hence, Gaussian shape is widely used; for example, the intermediate frequency tilters in spectrum analyzers are usually Gaussian. Interestingly enough, filters with Gaussian responses can be built by applying the central limit theorem to a cascade of filter stages. When a number of filters are cascaded, repeated convolution of the impulse responses results in a normal response [4].

Linear systems play a major role in electrical engineering: and Gaussian processes and linear systems go hand in hand with each other. For example. Gaussian process remains Gaussian after passage through a linear system. Also, when the underlying model is Gaussian, many optimal signal processing operations (mean square estimators, matched filter detectors, etc.) become linear. Furthermore, since Gaussian processes are completely described by second-order statistical properties, we understand why concepts such as power, signal-to-noise ratio, and spectral analysis dominate electrical engineering.

Some of the optimal properties of the normal distribution have important implications in the field of information theory. For example, entropy is a measure of randomness or uncertainty and hence measures the information content of the outcome of a random event. Of all probability distributions with fixed variance, it is the Gaussian function that maximizes relative or differential entropy. (This is a direct consequence of the fact that the logarithm of the Gaussian distribution is a quadratic function.) This property can be interpreted to mean that Gaussian noise is the most random or "noisiest" waveform [5. p. 281].

While the Gaussian model obviously results in the "best" in some situations, it can also represent the "worst" case many times and still be useful! For example, another result of information theory is that Gaussian noise yields the smallest channel capacity among all channels with additive noise having fixed power. Since the capacity measure restricts information rates, this result points to Gaussian noise as being the worst impairment a channel can have. Code design under such adverse conditions represents a conservative approach and yields satisfactory results [5, p. 245].

When I sent my friend Dr. O. Lord, language expert and scholar, a preliminary version of this column, he sent back a critical review. l reproduce below some of his less scathing remarks:

"While your column is mildly interesting from a scientific perspective, you seem to have completely ignored the religious and mystical connection. Don't forget what Albert Einstein said: 'Religion without science is blind. Science without religion is lame.' You seem to have missed the significance behind the fact that the Gaussian pulse shape solves the Schrodinger wave equation (for a harmonic oscillator with parabolic potential well) that arises in quantum mechanics [6]. Perhaps you are unaware of the similarities between the insights provided by the study of quantum mechanics and those produced by the study of Oriental religions and mysticism. I suggest you at least read some popular books, such as [7-9].

"I have reason to believe that the importance of the bell-shaped curve and central-limit-theorem-type arguments have been known to mankind much earlier than their discovery in the realm of science. As you know, bells have always played an important role in the rituals of many religions. (Remember *Bell*, *Book* and *Candle*?) Consecrated bells were believed to be able to disperse storms and pestilence, drive away devils, and extinguish fire [10]. I have an ancient manuscript written by a leader of o a mystical group of bell worshippers. In it, the author expounds upon the benefits of meditating on the bell shape. Commenting upon the time-frequency invariance of the bell shape, the author claims that meditation upon such a shape will enable a person to achieve, in both physical shape and mental plane, a state of invariance under space-time transformations as well as under the more mundane emotional turmoils one undergoes in daily life. The manuscript also describes in minute detail many mystical rituals involving bell shaped hair-pieces, ringing bells, and flowers arranged in bell-shaped mounds.

"Furthermore, I am at present trying to decipher another old manuscript wherein the anonymous author applies a central-limit-type of logical argument to develop a principle that can be loosely translated in modern terms as 'God is Gaussian.' Arguing that various religions are essentially independent and that the sum total of all such religions results in the correct concept of God, he applies a limiting type of formalism to describe God as essentially 'bell-shaped' in a conceptual plane.

"Thus I suggest that. instead of taking up trivial scientific pursuits such as the one in your column, you undertake important explorations showing the mystical connections between Gaussianity and the Quantum Reality that surrounds us."

Well, what can I say? I'll let you be the judge!

References

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