A Brief Review of Array Theory[1]

1 Review of Terminology

An antenna is a device that radiates energy due to an applied source excitation or receives radiation due to an external electromagnetic field. In the transmit mode, an excitation applied to the antenna terminals causes a current on the antenna. This current radiates, inducing electromagnetic fields in the region surrounding the antenna. The EM fields are easiest to analyze in the *far-field* where angular variation of the radiation fields are independent of the radial distance. While there is no theoretical definition of the far field region, by common definition, the far field is the region beyond radial distance R, where R satisfies

$$\begin{aligned} R &> D, \\ R &> \lambda, \\ R &> 2D^2/\lambda, \end{aligned}$$

where D is the largest dimension of the array. In this region, the radial component of the electric and magnetic fields are negligible and only the θ and ϕ components matter. In general, especially in wireless communications, antennas and wavelengths are small enough that we can safely assume the far-field condition holds.

Some terminology with regard to antennas:

- *Radiation Intensity*: The real power density in W/m^2 at any point is given by

$$\mathbf{W}_{rad} = \Re\left\{\mathbf{E} \times \mathbf{H}^*\right\}. \tag{1}$$

  Note that Balanis uses peak values, while we are using RMS values - he has a factor of 1/2 that is missing from our equation. In the far-field region this vector is pointed radially only, $\mathbf{W}_{rad} = W_{rad}\hat{\mathbf{a}}_r$ where W_{rad}, in line-of-sight conditions, is inversely proportional to r^2 (r is the radial distance). To obtain a measure independent of radial distance, the radiation intensity U, measured in (W/unit solid angle) and power radiated (P_{rad}) are given by

$$U(\theta, \phi) = r^2 W_{rad} \tag{2}$$

$$P_{\text{rad}} = \oint_S U(\theta,\phi) d\Omega = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta d\theta d\phi. \tag{3}$$

[1] Most of the material in this review comes from Balanis [1].

- *Directivity* : A measure of how directionally selective an antenna is. It is defined as the ratio of radiation intensity to average radiation intensity, U_0,

$$U_0 = \frac{1}{4\pi}\int_\theta\int_\phi U(\theta,\phi)d\phi d\theta = \frac{P_{\rm rad}}{4\pi} \tag{4}$$

$$D(\theta,\phi) = \frac{U(\theta,\phi)}{U_0} = \frac{4\pi U(\theta,\phi)}{P_{rad}}, \tag{5}$$

  If the direction (θ,ϕ) are not specified, the direction of maximum radiation intensity is assumed, i.e., maximum directivity. One may consider this to be the *directivity of the antenna.* Directivity is usually written in dB as D (in dB) $= 10\log_{10} D$.

- *Gain*: Also a measure of how directionally selective an antenna is. However, gain also accounts for the fact that some energy is lost in the antenna itself.

$$G(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_{in}}, \tag{6}$$

$$= \eta_{cd} D(\theta,\phi), \tag{7}$$

  where, P_{in} is the power input to the antenna and η_{cd} is the radiation efficiency of the antenna. Again, if the direction is not specified, maximum gain is assumed. Gain is also usually written in dB.

  Example

  $U = B_0 \sin\theta \sin^2\phi$ (W/unit solid angle), $0 < \theta < \pi$, $0 < \phi < 2\pi$. Radiation efficiency of antenna = 90%. Find directivity and gain.

  Answer

$$\begin{aligned}
U_0 &= \frac{P_{rad}}{4\pi} \\
&= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta d\theta d\phi = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} B_0\sin\theta\sin^2\phi\sin\theta d\theta d\phi \\
&= \frac{1}{4\pi}B_0\int_0^{\pi}\sin^2\theta d\theta\int_0^{2\pi}\sin^2\phi d\phi \\
&= \frac{1}{4\pi}B_0\frac{\pi}{2}\pi = B_0\frac{\pi}{8} \\
D(\theta,\phi) &= \frac{U(\theta,\phi)}{U_0} = \frac{8B_0\sin\theta\sin^2\phi}{B_0\pi} = \frac{8\sin\theta\sin^2\phi}{\pi} \\
D_{max} &= \frac{8}{\pi} = 2.54 = 4.0\text{dB} \\
G_{max} &= 0.9\times D_{max} = 0.9\times\frac{8}{\pi} = 2.29 = 3.6\text{dB}.
\end{aligned}$$

- *Radiation Pattern:* A plot of $U(\theta, \phi)$ or $G(\theta, \phi)$ versus θ and/or ϕ. However, in this class, we will focus on the azimuthal (ϕ) plane, assuming $\theta = \pi/2$. Figure 1 plots a sample beampattern.

- *Isotropic Radiator*: An *isotropic* radiator is one that radiates (receives) equally in (from) all directions - the field is constant, independent of θ and ϕ, i.e., $U(\theta, \phi) = U_0$. This idealized antenna, represented as a point sensor without directional information, is taken as the reference to express directive properties of more realistic antennas. Given an excitation I on an isotropic radiator the radiated field is (within a constant)

$$E = I\frac{e^{-jkr}}{r}. \tag{8}$$

  Note that the isotropic radiator is an idealized abstraction of a real world antenna - the field here, therefore, is not the physical electrical or magnetic field, but some field that can be measured.

  Importantly, *the signal processing community mostly assumes this kind of idealized antenna.* While it is almost unfair to do this, introducing electromagnetic analysis into signal processing usually results in unnecessary complications. Additionally, we now have a reasonably good approach that accounts for the electromagnetic behavior of more complicated antennas [2]. In this course, we will consistently assume isotropic radiators, unless specifically indicated otherwise. This is NOT to say that EM issues are not important, just that they can be addressed.

- *Directional Radiator*: A *directional* antenna transmits (receives) waves more effectively in (from) certain directions.

- *Omnidirectional Radiator*: An *omnidirectional* antenna is isotropic in one plane, but directive in another, i.e., it is a special type of directional antenna. Important examples include dipoles - a dipole lying along the z-axis radiates isotropically in the azimuth (ϕ) plane but directionally along the elevation (θ) plane.

- *Lobes*: Figure 1 plots a sample beampattern. A lobe is a portion of the radiation pattern surrounded by regions of relatively weak intensity.

  - The *major* or *mainlobe* or *mainbeam* is the lobe with the maximum radiation intensity.
  - A *minor* lobe is any lobe other than the main lobe.
  - A *sidelobe* is a lobe in any direction other than the intended direction.
  - A *backlobe* is a radiation lobe whose axis is approximately π with respect to the antenna axis (usually the mainlobe).

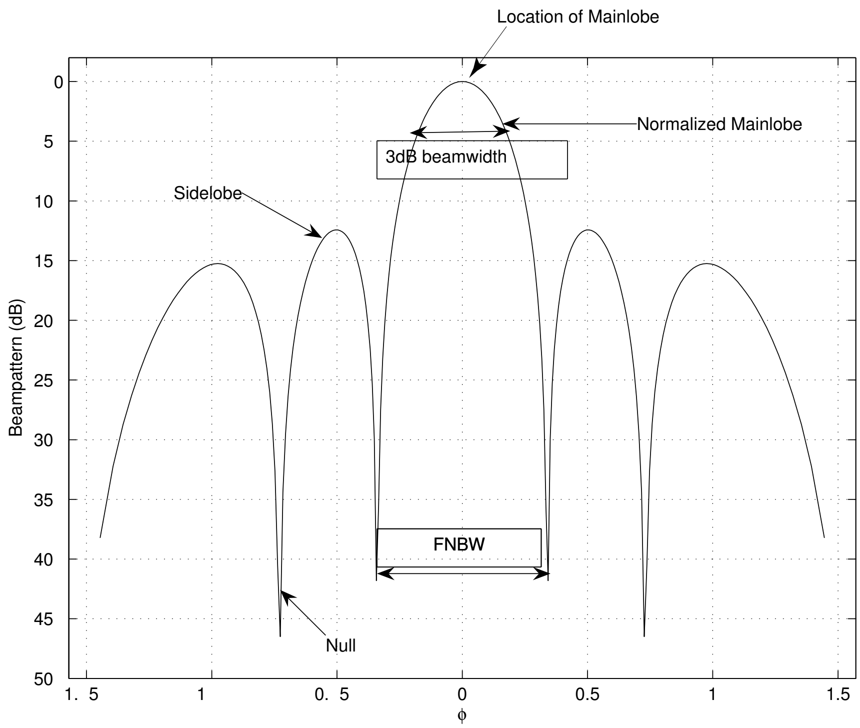

Figure 1: A sample beampattern

– A *grating lobe* is one where the radiation intensity is the same as the mainlobe, however, it is in an unwanted direction. Grating lobes arise when using arrays with wide inter-element spacing.

– *3dB (Half Power)/First Null Beamwidth*: The half power beamwidth is also known as 3dB beamwidth. It is the difference between the angles on either side of the "main" direction, at which the radiation intensity is half the maximum radiation intensity. The first null beamwidth is the difference in angle between the first nulls on either side of the "main" direction. The 3dB beamwidth is often approximated as half the first null beamwidth.

– *Bandwidth*: The range of frequencies over which some interesting antenna characteristic is within a chosen tolerance (usually 3dB). Interesting antenna characteristics include: input impedance (*impedance bandwidth*) and beam pattern (*pattern bandwidth*).

– *Polarization*: The antenna polarization is the same as polarization of the transmitted/received wave. The polarization of a wave is time varying direction and relative magnitude of the E-field, observed along the direction of propagation. Polarization can be linear (always along a straight line), circular (two orthogonal components of equal magnitude with a phase difference of $n\pi/2$, n integer) or elliptical (the general case).

On receive, an antenna may be able to receive more effectively from one polarization and less from another. In this case, there is a loss factor if the wave polarization does not match that of the antenna. In addition, since orthogonal polarizations do not interact, different streams of data may be transmitted on different polarizations.

- *Pattern Nulls*: Angles (usually discrete) at which the radiation pattern is zero or essentially zero. It is sometimes desirable, and possible, to obtain a broad null in the beam pattern.

  The goal of beamforming is to place pattern nulls in direction of the interference while simultaneously steering a mainbeam toward the "desired" user. Note that grating lobes, sidelobes and backlobes usually represent wasted energy.

2 Array Theory

This course deals with how to exploit *multiple antenna elements* also known as an *array*.

Figure 2 illustrates an array of isotropic elements placed along the x-axis. The spacing between the elements is a constant d. This is an example of a *linear, equispaced, array of isotropic point sensors*. In general, we will use this kind of array throughout this course.

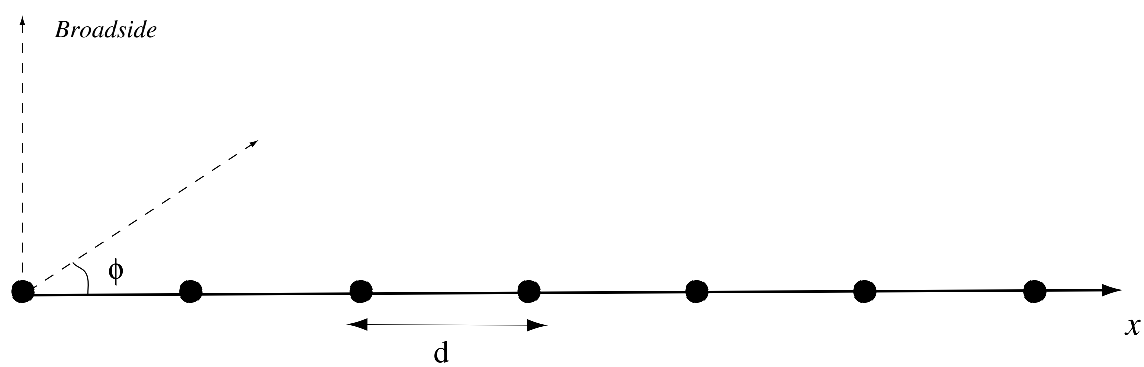

Figure 2: A linear array of equidistant elements

Consider the two element array in Fig. 3. The distance between the elements is d. The distances to a far field point from element 0 and element 1 are r_0 and r_1 while the excitations on the elements are I_0 and I_1 respectively. Using Eqn. (8), the total field is

$$\textit{Field} = I_0 \frac{e^{-jkr_0}}{r_0} + I_1 \frac{e^{-jkr_1}}{r_1}. \tag{9}$$

Because the field point is in the far field, the vectors from elements 0 and 1 are approximately parallel, leading to

$$r_1 = r_0 - d\cos\phi = r - d\cos\phi, \tag{10}$$

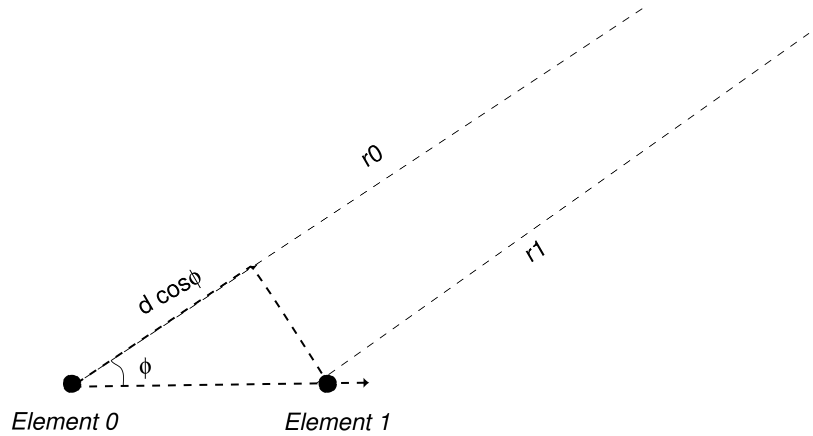

Figure 3: An array of two elements

where ϕ is the azimuthal angle of the far field point. Also, $d\cos\phi << r_0$. We can ignore the small difference between r_0 and r_1 in the denominator of Eqn. (9), however, this difference is important in the numerator since the exponent makes it a phase term. Therefore, setting $r = r_0$,

$$Field = \frac{e^{-jkr}}{r}\left[I_0 + I_1 e^{jkd\cos\phi}\right]. \tag{11}$$

The leading term e^{-jkr}/r is constant for a given radial distance and is irrelevant for directivity/gain. Generalizing this to an array of N elements, and ignoring the constant leading term, we get the *array factor*

$$\begin{aligned} AF &= \left[I_0 + I_1 e^{jkd\cos\phi} \ldots I_{N-1} e^{jkd(N-1)\cos\phi}\right], \\ &= \sum_{n=0}^{N-1} I_n e^{jnkd\cos\phi} = \sum_{n=0}^{N-1} I_n e^{jn\psi}, \end{aligned} \tag{12}$$

where $\psi = kd\cos\phi$.

The array factor (or at least its magnitude squared) is also the beampattern for an array of isotropic sensors. For other types of array elements, the overall beampattern can be approximated as a product of the array factor and an "element factor", the beampattern of an individual element. Since we are using isotropic sensors, we will use the "array factor" "beampattern" interchangeably.

Note that the array factor is the *Discrete Time Fourier Transform* of the element excitations. The variable ψ is called *spatial frequency.* Many signal processing algorithms assume this Fourier relationship between the beampattern and the excitation. This relationship also allows us to derive many array properties that arise from Fourier theory. In fact, this Fourier relationship is not exclusive to arrays. The far-field of an antenna is always the Fourier transform of the excitation current.

We will consider in some detail the special case of

$$I_n = Ie^{jn\beta}, \tag{13}$$

which represents a linear phase front across the array. The excitation has a phase slope of β. The resulting array factor is given by

$$\begin{aligned} AF &= \sum_{n=0}^{N-1} I_n e^{jnkd\cos\phi}, \\ &= \sum_{n=0}^{N-1} I e^{jn(\beta + kd\cos\phi)}, \qquad (14) \\ &= \sum_{n=0}^{N-1} I e^{jn(\beta+\psi)}, \qquad (15) \\ &= I\frac{\left[1 - e^{jN(\beta+\psi)}\right]}{\left[1 - e^{j(\beta+\psi)}\right]}. \qquad (16) \end{aligned}$$

We can simplify the array factor to be

$$\begin{aligned} AF &= I\frac{e^{j\frac{N}{2}(\beta+\psi)}}{e^{j\frac{1}{2}(\beta+\psi)}}\left[\frac{e^{-j\frac{N}{2}(\beta+\psi)} - e^{j\frac{N}{2}(\beta+\psi)}}{e^{-j\frac{1}{2}(\beta+\psi)} - e^{j\frac{1}{2}(\beta+\psi)}}\right] \\ &= Ie^{j\frac{N-1}{2}(\beta+\psi)}\left[\frac{\sin\left(\frac{N(\beta+\psi)}{2}\right)}{\sin\left(\frac{(\beta+\psi)}{2}\right)}\right] \qquad (17) \end{aligned}$$

Our main goal is to evaluate the power beam pattern, i.e., only the magnitude of the function AF matters. Furthermore, parameters such as directivity are independent of constants. We will therefore focus on the functional form within the square brackets. Note that this is the famous $\sin/\sin$ form we have seen in DTFT theory.

We can make some important comments:

- *Maximum*: The maximum of this function occurs at $(\beta + \psi) = 0$ where it takes the value N. We will normalize this function and set

$$AF = \frac{1}{N}\left[\frac{\sin\left(\frac{N(\beta+\psi)}{2}\right)}{\sin\left(\frac{(\beta+\psi)}{2}\right)}\right]. \tag{18}$$

  with maximum value of unity.

- *Mainbeam Location*: The location of the mainbeam is determined by the equation $(\beta + \psi) = 0 \Rightarrow kd\cos\phi\ +\ \beta = 0$. The mainbeam is located at

$$\cos\phi_m = -\frac{\beta}{kd}. \tag{19}$$

This relationship is extremely important because it tells us that the location of the mainbeam can be controlled by the phase slope, β. Appropriately choosing β *steers* the mainbeam to any desired location. This is why such an array is called a *phased array.*

- *Nulls*: The nulls of the array factor occur at

$$\begin{aligned} \sin\left(\frac{N}{2}(\beta+\psi)\right) &= 0, \\ \Rightarrow \frac{N}{2}(\beta+\psi) &= m\pi; \quad m = 1, 2, \ldots N-1, \\ \Rightarrow kd\cos\phi_n + \beta &= \frac{2m\pi}{N}, \\ \Rightarrow (\cos\phi_n - \cos\phi_m) &= \frac{2m\pi}{Nkd} \end{aligned} \tag{20}$$

Note that the nulls are equally spaced in ψ (equivalently $\cos\phi$ space). Furthermore, Eqn. (20) (in conjunction with Eqn. (19)) tells us that the location of the mainbeam also determines the location of the nulls. Independent placement of nulls and the mainbeam (as we desire for interference cancellation) requires *adaptive beamforming*, a subject of future discussion.

- *Beamwidth*: The half power beamwidth is given by the point where $(AF)^2$ is equal to 1/2,

$$\begin{aligned} \frac{1}{N^2}\left[\frac{\sin\left(\frac{N(\beta+\psi)}{2}\right)}{\sin\left(\frac{(\beta+\psi)}{2}\right)}\right]^2 &= \frac{1}{2}, \\ \Rightarrow \frac{N(\beta+\psi)}{2} = \frac{N}{2}\left(kd\cos\phi_h + \beta\right) &= \pm 1.391, \end{aligned} \tag{21}$$

where ϕ_h represents the angle at which the radiated power is half the peak value. Using Eqns. (19) and (21), the beamwidth is given by

$$\cos\phi_h - \cos\phi_m = \frac{\pm 1.391 \times 2}{Nkd}. \tag{22}$$

There are two points to be made with regard to Eqn. (22). First, as expected the beamwidth decreases with increasing values of N or d. It is the total length of the array $(N-1)d$ that matters. Furthermore, $k = 2\pi/\lambda$, i.e., it is the *total array length in terms of wavelength* that determines the beamwidth. There is no free lunch - a narrow beamwidth is desirable but must be paid for in terms of larger antenna arrays.

Second, in $\cos\phi$ space, the beamwidth is independent of the location of the mainbeam. However, since $\cos\phi$ has the greatest slope near $\cos\phi = 0$ or broadside (at this point, the derivative of $\cos\phi$, which is $\sin\phi$, is maximum), the beamwidth in ϕ space *increases* with *increasing* value of $|\cos\phi_m|$. This implies that the beamwidth increases as the array is scanned off broadside. This result comes about due to the non-linear relationship between ϕ and ψ.

- *Sidelobe Level*: The first sidelobe occurs at approximately

$$\frac{N(\beta + \psi_{sl})}{2} = \frac{3\pi}{2}. \tag{23}$$

The numerator in Eqn. (18) is unity and hence,

$$\Rightarrow AF_{sl} \simeq \frac{1}{N(\beta + \psi_{sl})/2} = \frac{2}{3\pi} = 0.212$$

$$AF_{sl}(\mathrm{dB}) = 20\log(AF_{\mathrm{sl}}) = -13.5\mathrm{dB}. \tag{24}$$

A curious conclusion is that this, rather high, sidelobe level is independent of the number of elements, the size of the array or other factors under our control. As can be expected, this -13dB figure also arises when designing bandpass filters.

Note that the above discussion of antenna parameters like beamwidth, directivity, sidelobes and location of nulls is only for the assumed excitation of $I_n = Ie^{jn\beta}$, a constant multiplied with a linear phase change. These parameters can be controlled using *windows*. Harris [3] provides a detailed description of several windows and their associated parameters.

- *Visible Region*: The DTFT relationship in Eqn. (15) implies that the beam pattern is periodic in ψ, with period 2π, as shown in Fig. 4 for a 11-element array[2]. However, real angles are restricted to $0 \leq \phi \leq \pi$. Therefore, of Fig. 4, only a certain section can be "seen" in real space. This region is set by the range of $-1 \leq \cos\phi \leq 1$ in the definition of ψ,

$$-kd \leq \psi \leq kd. \tag{25}$$

This region is the *visible* region corresponding to a physically measurable sector of the spatial frequency, ψ.

Let us examine Fig. 4 for some candidate choices of d and β. We choose:

  - $d = \lambda/8, kd = \pi/4, \beta = 0$: The visible region is given by $-\pi/4 \leq \psi \leq \pi/4$. The array factor as a function of angle ϕ is plotted in Fig. 5. Note the extremely broad beam pattern due the small array length.
  - $d = \lambda/4, kd = \pi/2, \beta = 0$: The visible region is given by $-\pi/2 \leq \psi \leq \pi/2$. The array factor as a function of angle ϕ is plotted in Fig. 6. With this larger array spacing, the mainbeam is narrower and sidelobes appear. Note that, as expected, the highest sidelobe is at approximately 0.212 or -13dB down from the mainlobe.
  - $d = \lambda/4, kd = \pi/2, \beta = \pi/4$: The visible region is given by $-\pi/4 \leq \psi \leq 3\pi/4$. The array factor as a function of angle ϕ is plotted in Fig. 7. As compared to Fig. 6, the mainbeam is shifted to the location $\cos\phi = -\beta/kd$ or $\phi = 2.09$ radians.

[2] Note that the plot is not in dB - this was done to help visualize the "visible region" and is rather unusual

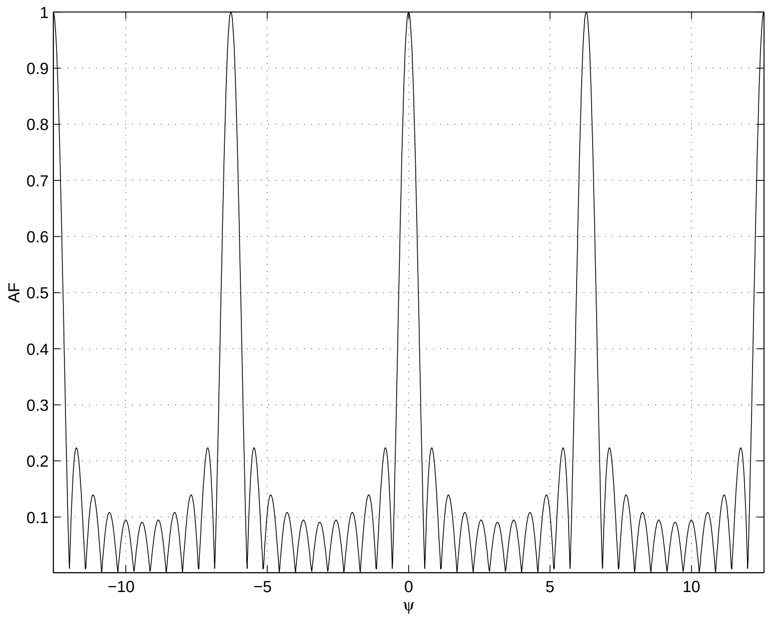

Figure 4: Array factor versus ψ. N = 11.

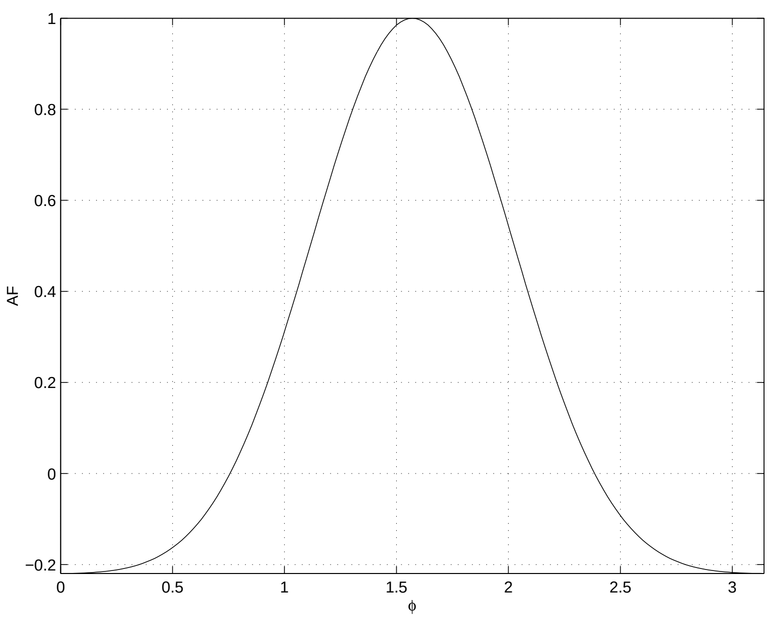

Figure 5: Array factor versus ϕ. $N = 11, d = \lambda/8, \beta = 0$

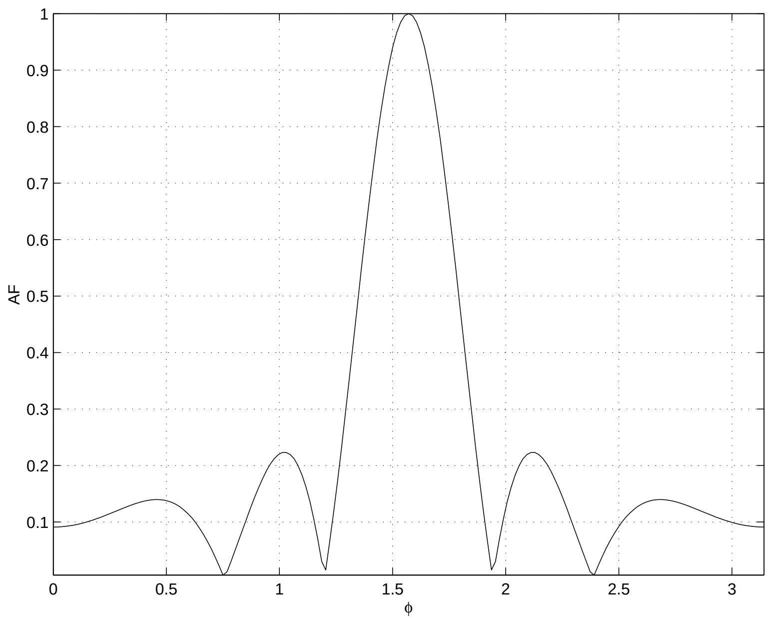

Figure 6: Array factor versus ϕ. $N = 11, d = \lambda/4, \beta = 0$.

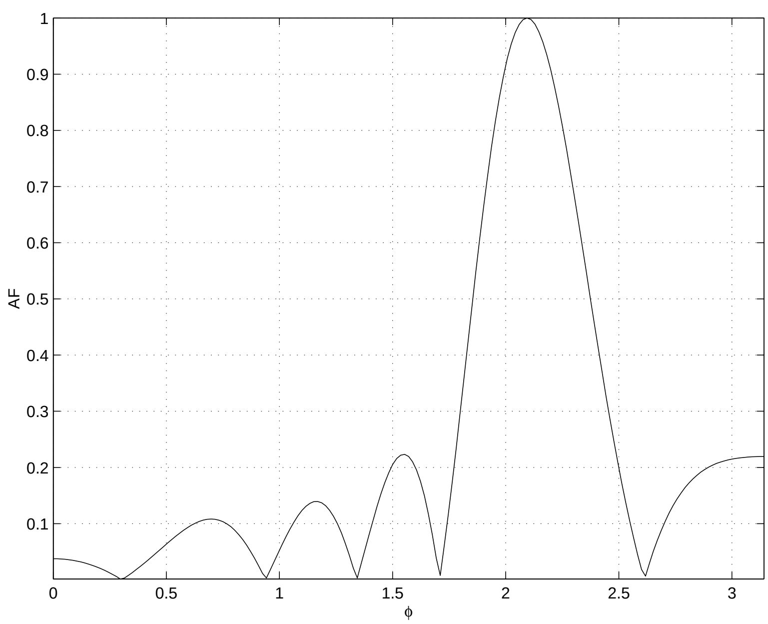

Figure 7: Array factor versus ϕ. $N = 11, d = \lambda/4, \beta = \pi/4$.

- $d = \lambda/2, kd = \pi, \beta = 0$: The visible region is given by $-\pi \leq \psi \leq \pi$, i.e., one complete period of ψ. The array factor as a function of angle ϕ is plotted in Fig. 8. Note that the mainbeam is narrower than in Fig. 6.
- $d = \lambda/2, kd = \pi, \beta = \pi/2$: The visible region is given by $-\pi/2 \leq \psi \leq 3\pi/2$. The array factor as a function of angle ϕ is plotted in Fig. 9. As compared to Fig. 8, the mainbeam is shifted to the location $\cos\phi = -\beta/kd$.
- $d = \lambda/2, kd = \pi, \beta = \pi$: The visible region is given by $0 \leq \psi \leq 2\pi$. The array factor as a function of angle ϕ is plotted in Fig. 10. Due to this choice of β, the mainbeam is supposed to be located at $\cos\phi = -\pi/\pi = -1 \Rightarrow \phi = \pi$. However, due to the periodicity of AF as a function of ψ, there is a beam, as strong as the mainbeam, in direction $\phi = 0$. This extra *grating lobe* arises due to the scanning of the antenna array (moving the visible region in ψ space).

  The grating lobe represents a significant waste of energy. It arises is arrays that are reasonably widely spaced and scanned. For this reason, an array with $\lambda/2$ spacing would not be scanned beyond approximately 60^o.
- $d = 2\lambda, kd = 4\pi, \beta = 0$: The visible region is given by $-4\pi \leq \psi \leq 4\pi$. The array factor as a function of angle ϕ is plotted in Fig. 11. Note that even though this beam is not scanned, grating lobes still occur. This is due to the wide array spacing setting the visible region to several periods of the array function. If an array has spacing $d \geq \lambda$, grating lobes always occur, independent of scan angle.

  Earlier we had shown that to achieve greater directivity one needs a long array (increase the total array length $(N-1)d$. This discussion on grating lobes shows why one cannot arbitrarily separate array elements without penalty. Yes, one can separate antennas and get a narrow mainbeam. However, separate them too far and you end up with grating lobes and waste energy.

A final note: in deriving the array factor as a function of ϕ in Eqn. (15), we assumed that the elevation angle was set to $\pi/2$. For any other elevation angle (θ),

$$r_{n+1} = r_n - d\cos\phi\sin\theta.$$

Therefore, in all our equations $\cos\phi$ is replaced with $\cos\phi\sin\theta$. It is common to assume $\theta = \pi/2$ for convenience. In a communication system, given the large separation between base station and mobiles, all mobiles are within a narrow section of elevation angle. Setting the elevation angle to a constant is, therefore, a fairly good approximation.

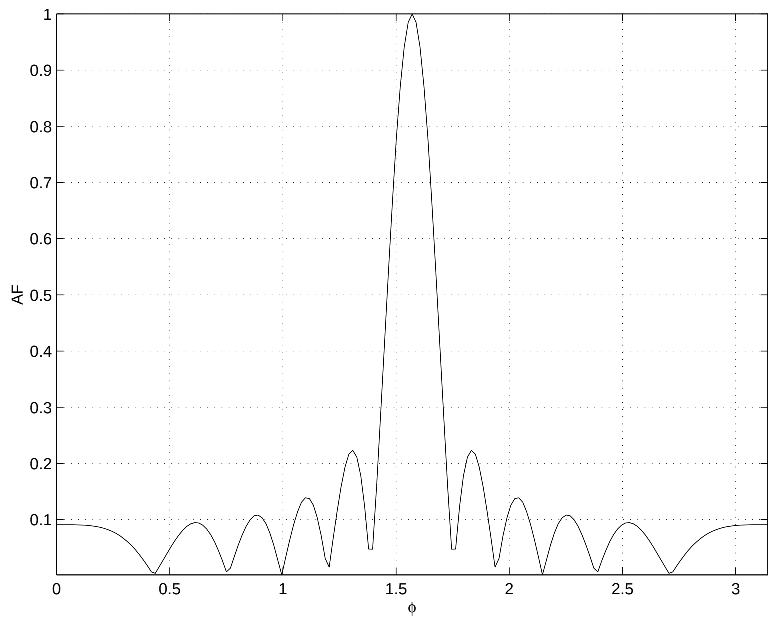

Figure 8: Array factor versus ϕ. $N = 11, d = \lambda/2, \beta = 0$.

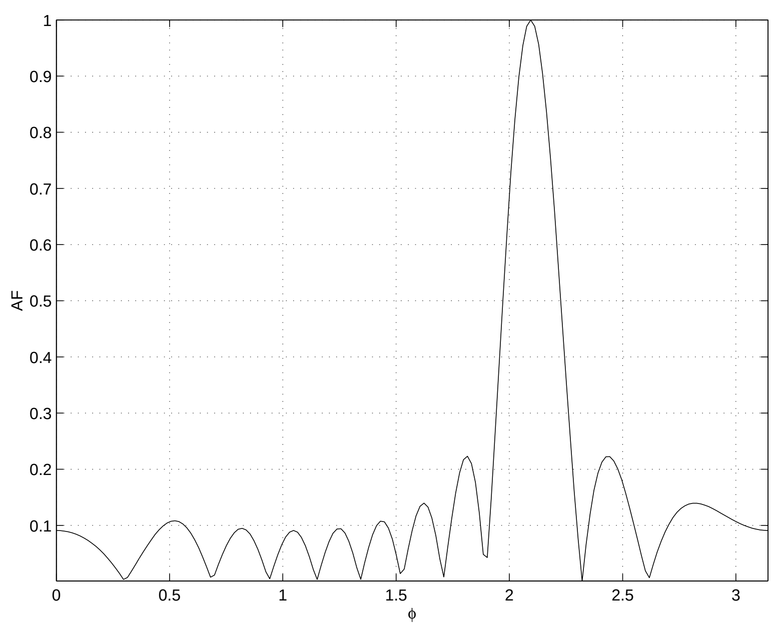

Figure 9: Array factor versus ϕ. $N = 11, d = \lambda/2, \beta = \pi/2$.

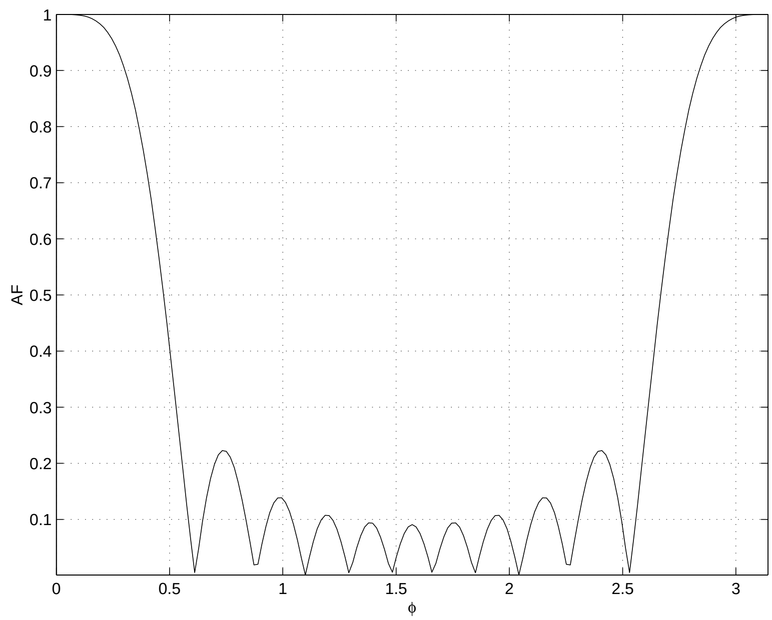

Figure 10: Array factor versus ϕ. $N = 11, d = \lambda/2, \beta = \pi$.

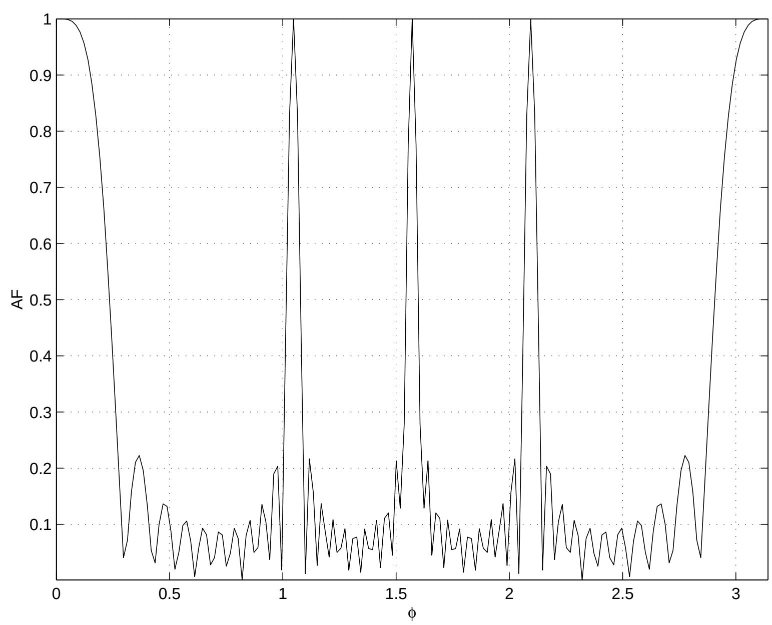

Figure 11: Array factor versus ϕ. $N = 11, d = 2\lambda, \beta = 0$.

3 Pattern Synthesis using Fourier Methods

So far we have answered the question, "Given the current on the elements, what is the radiation pattern?". *Pattern synthesis* reverses this question - we ask "Given a desired power pattern, what excitation do we require?". We will investigate the simplest answer - Fourier Synthesis.

We have defined the array factor AF as

$$AF = \sum_{n=0}^{N-1} I_n e^{jn\psi}.$$

Using the inverse DTFT, the solution for I_n is straightforward

$$I_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} AF(\psi) e^{-jn\psi} d\psi. \tag{26}$$

The problem here is that this equation assumes the range of n to be $-\infty < n < \infty$. In the case of an array, only N of these may be used. Limiting the inverse DTFT to N samples introduces an implicit *windowing* to the inverse DTFT samples (a rectangular window that is non-zero between $n = 0$ and $n = N - 1$ only. This introduces an effective *discontinuity* between I_{-1} and I_0. From Fourier theory, a discontinuity is the source of Gibbs' phenomenon and other highly undesirable occurrences. This is particularly true as the array factor is defined in terms of its magnitude, i.e., the desired pattern is given to be real. The IDTFT leads to a symmetric solution ($I_{-n} = I_n^*$). For this reason, the reference point of the array is moved from element 0 to the center. The window is symmetric and the discontinuity is significantly reduced.

For an array with an odd number of elements, $N = 2P + 1$, the array reference will be at the center element, with elements now numbered $n = -P, -P + 1, \ldots, -1, 0, 1, \ldots, P - 1, P$, with locations $x_n = nd$. For an array with an even number of elements $N = 2P$, the array reference will be in between the center two elements, with elements numbered $n = -P, -P+1, \ldots, -1, 1, \ldots, P-1, P$. The locations are given by $x_n = (2n+1)d/2$, $-P \leq n \leq -1$ and $x_n = (2n-1)d/2$, $1 \leq n \leq P$. We shall look into the simpler case of an array with an odd number of elements.

To find the required current on each element, we use Eqn. (26), with $-P \leq n \leq P$. Note that even in this case, the IDTFT equation assumes an infinite array. However, in most cases, the unwanted phenomena resulting from truncating the array is significantly reduced. A simple example is a rectangular array function,

$$\begin{aligned} AF &= 1, \quad |\psi| \leq \pi/\sqrt{2} \\ &= 0, \quad |\psi| > \pi/\sqrt{2} \end{aligned}$$

which results in

$$I_n = \frac{1}{\sqrt{2}}\left[\frac{\sin(n\pi/\sqrt{2})}{n\pi/\sqrt{2}}\right]$$

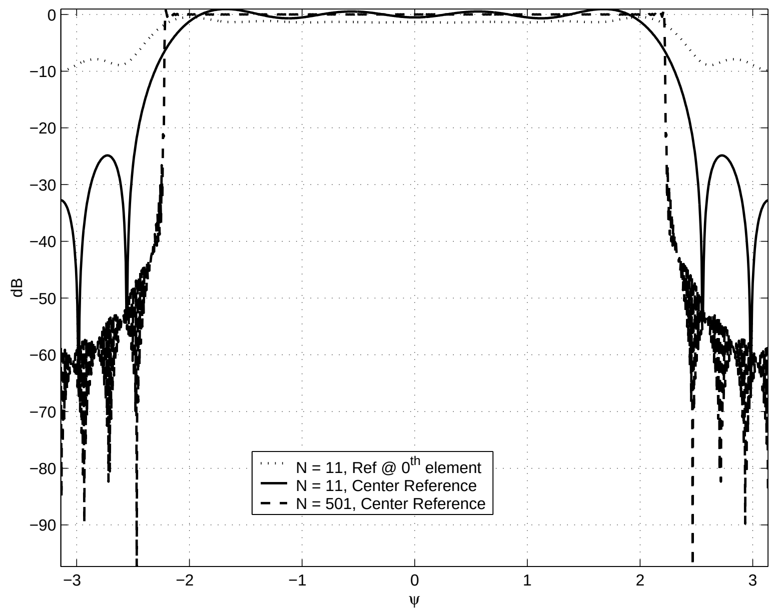

Figure 12: The resulting beampattern from three Fourier syntheses

This equation is true independent of whether we choose the center reference or the reference at the 0^{th} element.

Figure 12 plots the resulting beampattern from three cases: $N = 11$, using the reference at the 0^{th} element, $N = 11$, with center reference and $N = 501$, center reference. Clearly the first case produces an essentially useless beampattern. This is because of the large "discontinuity" between the 0^{th} and the (missing) $(-1)^{th}$ element.

With N still equal to 11, the lower sidelobes are clearly visible when the reference is at the center. Also, as expected, when the array comprises a huge number of elements ($N = 501$), a nearly ideal pattern is obtained. Note that the plot is a function of the array factor as a function of ψ, not ϕ (the physical angle). Hence, there has been no assumption made regarding the spacing between the elements or the scan angle. The improved sidelobes for the large value of N comes directly from Fourier theory.

Scanning the beam: Often we require several beams with the same beam shape, but centered at several angles, i.e., the beam must be scanned. The procedure is to design the beam centered at $\phi = \pi/2$ (broadside) and then introduce a phase shift (β) to scan the beam to the desired beam location¿ The first step is to given AF centered at $\psi = 0$, find a set of currents I_n. Then the beam centered at ϕ is obtained from the currents $I_n e^{j\beta}$, $\beta = -kd \cos \phi$. Choosing multiple values of β results in a multiple set of beams.

4 Arrays on Receive

So far we have focused on transmitting arrays. In this section we will develop the relevant terminology associated with arrays receiving incident signals. We will assume that the signals arise from the far field, propagating as a *plane wave*. We start with *steering vectors*.

Definition: The steering vector, associated with an angle ϕ, is the length-N vector of the array responses due to an incident field from angle ϕ.

Consider the two-element array in Fig. 3. This array is now receiving a signal from direction ϕ. The signal arrives at element #1 $\Delta t = d\cos\phi/c$ seconds before it arrives at element #0. Since we are dealing with narrowband signals at frequency f, this time advance is equivalent to a phase shift of $e^{j2\pi f\Delta t} = e^{j2\pi f d\cos\phi/c} = e^{j2\pi d\cos\phi/\lambda} = e^{jkd\cos\phi}$. The signal at element #1 (s_1) and the signal at element #0 (s_0) are related by

$$s_1 = s_0 e^{jkd\cos\phi}.$$

The two-element steering vector is therefore,

$$\mathbf{s}(\phi) = s_0 \begin{bmatrix} 1 & e^{jkd\cos\phi} \end{bmatrix}^T.$$

The signal s_0 is a constant across the steering vector. This constant can be safely ignored as the essential information relating the signal direction and the steering vector is in the vector

$$\mathbf{s}(\phi) = \begin{bmatrix} 1 & e^{jkd\cos\phi} \end{bmatrix}^T. \tag{27}$$

Generalizing this to a N-element array of isotropic, equispaced, point sensors, the steering vector associated with an angle ϕ is

$$\mathbf{s}(\phi) = \begin{bmatrix} 1 \; e^{jkd\cos\phi} \; e^{2jkd\cos\phi} \; \ldots \; e^{j(N-1)kd\cos\phi} \end{bmatrix}^T. \tag{28}$$

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