Receive Diversity [1]

1 Introduction

When discussing beamforming we used the resources of an antenna array to service multiple users simultaneously. The interference cancellation schemes were formulated independent of whether the channels were line-of-sight (LOS) or fading. Diversity combining devotes the entire resources of the array to *service a single user*. Specifically, diversity schemes enhance *reliability* by minimizing the channel fluctuations due to fading. The central idea in diversity[2] is that different antennas receive different versions of the same signal. The chances of all these copies being in a deep fade is small. These schemes therefore make most sense when the fading is independent from element to element and are of limited use (beyond increasing the SNR) if perfectly correlated (such as in LOS conditions). Independent fading would arise in a dense urban environment where the several multipath components add up very differently at each element.[3].

Early in this course we saw that fading has three components: path loss, large-scale and small-scale fading. Over fairly long periods the first two components are approximately constant and can be dealt with using power control. Furthermore, these components of fading are very close to being constant across all elements of the array (perfectly correlated). Diversity combining is specifically targeted to counteract small scale fading. We will therefore use slow, flat, Rayleigh fading as our model for the signal fluctuation. It must be emphasized that the Rayleigh model is the easiest and most tractable. However, this model is not valid in all situations. The results we present here will therefore be only "ballpark" figures, to illustrate the workings of diversity combining.

The physical model assumes the fading to be independent from one element to the next. Each element, therefore, acts as an independent sample of the random fading process (here Rayleigh), i.e., each element of the array receives an independent copy of the transmitted signal. Our goal here is to combine these independent samples to achieve the desired goal of increasing the SNR and reducing the BER. Diversity "works" because given N elements in the receiving antenna array we receive N independent copies of the same signal. It is unlikely that all N elements are in a deep fade. If at least one copy has reasonable power, one should conceivably be able to adequately process the signal. We emphasize that the model developed below is for a *single user*.

[1]Most of the material in this review is from Janaswamy [1]. This material also appears in the book by Godara [2]

[2]Diversity arises in various forms - time, frequency, spatial (our case), etc.

[3]Saying two random variables are *independent* is a far stronger statement than saying they are *uncorrelated*

2 The Model

Consider a single-user system model wherein the received signal is a sum of the desired signal and noise:

$$\mathbf{x} = \mathbf{h}u(t) + \mathbf{n} \tag{1}$$

where $u(t)$ is the unit power signal transmitted, $\mathbf{h}$ represents the channel (including the signal power) and $\mathbf{n}$ the noise. The power in the signal over a single symbol period, T_s, at element n, is

$$P = \frac{1}{T_s}\int_0^{T_s} |h_n(t)|^2|u(t)|^2 dt = |h_n(t)|^2 \frac{1}{T_s}\int_0^{T_s} |u(t)|^2 dt = |h_n|^2, \tag{2}$$

where, since we are assuming slow fading, the term $|h_n(t)|$ remains constant over a symbol period and can be brought out of the integral and $u(t)$ is assumed to have unit power. Setting $\mathrm{E}\{|n_n(t)|^2\} = \sigma^2$ and we get the instantaneous SNR at the n-th element (γ_n) to be

$$\gamma_n = \frac{|h_n|^2}{\sigma^2}. \tag{3}$$

This instantaneous SNR is a random variable with a specific realization given the channel realization h_n. The expectation value taken to estimate the noise power is therefore taken over a relatively short time period. Later on we will also find a long-term average SNR.

We are assuming Rayleigh fading, so $h_n = |h_n|e^{j\angle h_n}$, where $\angle h_n$ is uniform in $[0, 2\pi)$ and $|h_n|$ has a Rayleigh pdf, implying $|h_n|^2$ (and γ_n) has an exponential pdf

$$|h_n| \sim \frac{2|h_n|}{P_0} e^{-|h_n|^2/P_0}, \tag{4}$$

$$\gamma_n \sim \frac{1}{\Gamma} e^{-\gamma_n/\Gamma}, \tag{5}$$

$$\Gamma = \mathrm{E}\{\gamma_n\} = \frac{E\{|h_n|^2\}}{\sigma^2} = \frac{P_0}{\sigma^2}. \tag{6}$$

The *instantaneous SNR at each element* that is an exponentially distributed random variable. Γ represents the average SNR at each element. This is also the SNR of a single element antenna, i.e., the SNR if there were no array. Γ will, therefore, serve as a baseline for the improvement in SNR.

2.1 Figures of Merit

In general we will use two figures of merit - outage probability and the bit error rate (BER) for BPSK modulation. In this section we develop these figures of merit for the single-input single-output (SISO) case.

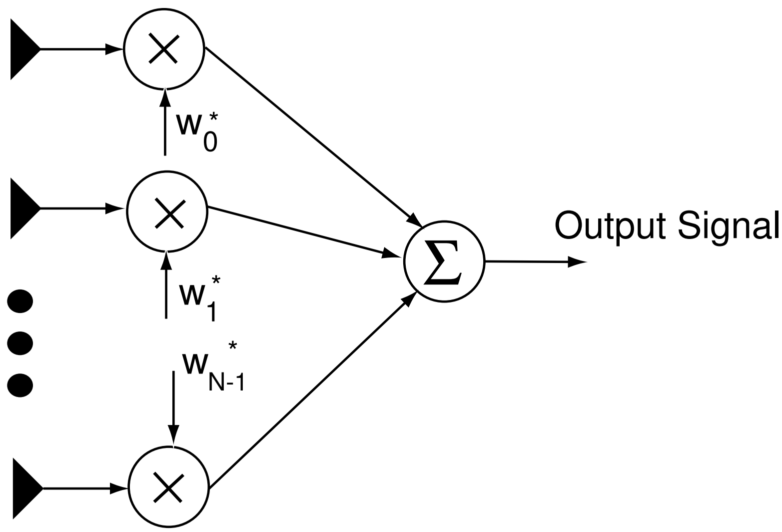

Figure 1: The receiver in a diversity combining system

The outage probability is defined as the probability that the output SNR, γ is below a threshold, γ_s. Since the SNR is exponentially distributed,

$$\begin{aligned} P_{\text{out}} &= P\left(\gamma < \gamma_s\right) = \int_0^{\gamma_s} \frac{1}{\Gamma} e^{-\gamma/\Gamma} d\gamma \\ &= \left[1 - e^{-\gamma_s/\Gamma}\right] \end{aligned} \tag{7}$$

Note that as $\Gamma \to \infty$, $P_{\text{out}} \propto 1/\Gamma$.

The bit error rate of a BPSK system *given a SNR of* γ is given by $\text{erfc}(\sqrt{2\gamma}) = Q\left(\frac{|h|}{\sigma}\right)$, where

$$Q(x) = (1/\sqrt{2\pi}) \int_x^{\infty} e^{-t^2/2} dt.$$

The BER averaged over the Rayleigh fading of Eqn. (4) is therefore

$$\begin{aligned} \text{BER} &= \int_0^{\infty} \frac{2|h|}{P_0} e^{-|h|^2/P_0} Q\left(\frac{|h|}{\sigma}\right) d(|h|) \\ &= \frac{1}{2}\left(1 - \sqrt{\frac{\Gamma}{1+\Gamma}}\right) \end{aligned} \tag{8}$$

The final equation uses Eqn. (3.61) of Verdu [3]. Note,

$$\lim_{\Gamma \to \infty} \text{BER} = \frac{1}{4\Gamma} \tag{9}$$

Using the BER and P_{out} expressions an important conclusion is that both these error expressions fall as (1/SNR) as SNR $\to \infty$. This extremely slow fall off in error rate is due to the *variance* in the SNR arising from the random channel.

Within diversity combining (or diversity reception) are three common techniques: Selection Combining, Maximal Ratio Combining (MRC) and Equal Gain Combining (EGC). For all three, the goal is to find a set of weights $\mathbf{w}$, as shown in Fig. 1. The structure is similar to what we used in developing interference cancellation. Here, however, the weights are chosen to minimize the impact of fading for a single user. The three techniques differ in how this weight vector is chosen. In all three cases we assume that *the receiver has the required knowledge of the channel fading vector* $\mathbf{h}$.

3 Selection Combining

Selection combining is to the elements what switched beamforming was to beams. As each element is an independent sample of the fading process, the element with the greatest SNR is chosen for further processing. In selection combining therefore,

$$w_k = \begin{cases} 1 & \gamma_k = \max_n\{\gamma_n\} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Since the element chosen is the one with the maximum SNR, the output SNR of the selection diversity scheme is $\gamma = \max_n\{\gamma_n\}$. Such a scheme would need only a measurement of signal power, phase shifters or variable gains are not required. To analyze such a system we look at the probability of outage, BER, and resulting improvement in SNR.

The probability of outage is the probability that the output SNR falls below a threshold γ_s, i.e., the SNR of all elements is below the threshold. Therefore,

$$\gamma = \max_n\{\gamma_n\}, \tag{11}$$

$$P_{\text{out}} = P[\gamma < \gamma_s] = P[\gamma_0, \gamma_2, \ldots \gamma_N < \gamma_s] = \prod_{n=0}^{N-1} P[\gamma_n < \gamma_s], \tag{12}$$

where the final product expression is valid because the fading at each element is assumed independent. This would not be true if we had only assumed the fading to be uncorrelated from one element to the next. Using the pdf of γ_n,

$$\begin{aligned} P[\gamma_n < \gamma_s] &= \int_0^{\gamma_s} f_\Gamma(\gamma_n)d\gamma_n = \int_0^{\gamma_s} \frac{1}{\Gamma}e^{-\gamma_n/\Gamma}d\gamma_n, \\ &= \left[1 - e^{-\gamma_s/\Gamma}\right] &(13) \\ \Rightarrow P_{\text{out}}(\gamma_s) &= \left[1 - e^{-\gamma_s/\Gamma}\right]^N. &(14) \end{aligned}$$

The outage probability therefore decreases *exponentially* with the number of elements. Figure 2 illustrates the improvement in outage probability as a function of the number of elements in the

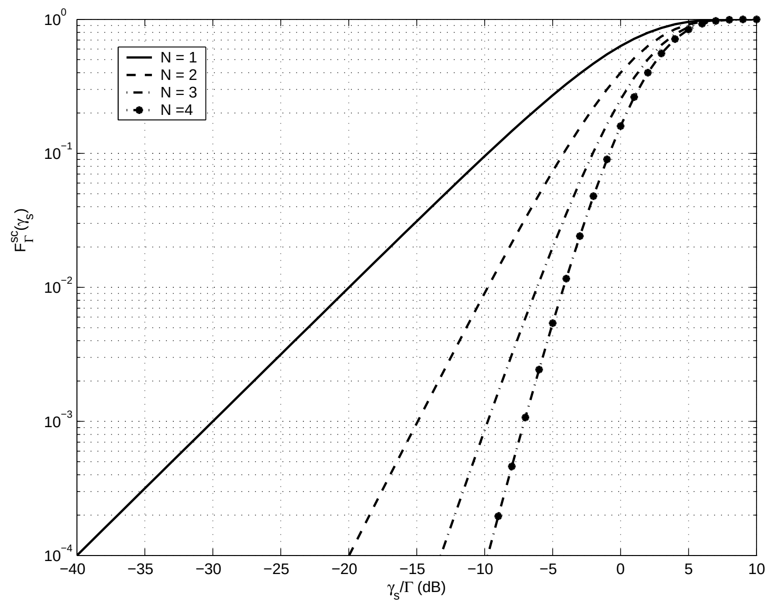

Figure 2: Performance of a selection combining system

array. As is clearly seen, selecting between just two elements results in significant performance improvements, almost 7dB at an outage probability of 1%. Note that if using two elements the output SNR can, at best, double. Note also the linear relationship (in the log-log plot) in Fig. 2 (expected given the exponential relationship between $P_{\rm out}$ and γ_s/Γ. The slope of the plot increases with increasing N. We will later define this notion rigorously as the diversity order.

$P_{\rm out}$ also represents the cdf of the output SNR as a function of the threshold γ_s. The pdf of the output SNR, γ, is therefore

$$f_\Gamma(\gamma) = \frac{dP_{\rm out}(\gamma)}{d\gamma} = \frac{N}{\Gamma} e^{-\gamma/\Gamma} \left[1 - e^{-\gamma/\Gamma}\right]^{N-1}. \tag{15}$$

At this point we have derived the probability of outage and the pdf of the output SNR. Two other possible figures of merit are the improvement in the average SNR (versus the average SNR at each element, Γ) and the improvement in BER. The average output SNR is

$$\begin{aligned} E\{\gamma\} &= \int_0^\infty \gamma f_\Gamma(\gamma) d\gamma = \int_0^\infty \gamma \frac{N}{\Gamma} e^{-\gamma/\Gamma} \left[1 - e^{-\gamma/\Gamma}\right]^{N-1} d\gamma, \\ &= \Gamma \sum_{n=1}^{N} \frac{1}{n}, & (16) \\ &\simeq \Gamma \left(C + \ln N + \frac{1}{2N} \right), & (17) \end{aligned}$$

where the final approximation is valid for relatively large values of N. C is Euler's constant. The improvement in SNR over that of a single element is of order of $(\ln N)$.

The overall error rate is obtained by integrating the conditional error rate at a given SNR. For BPSK modulation, the conditional bit error rate (BER) is $\mathrm{erfc}(\sqrt{2\gamma})$ and the overall error rate is

$$P_e = \int_0^\infty (\mathrm{BER}/\gamma)\, f_\Gamma(\gamma) d\gamma = \int_0^\infty \mathrm{erfc}(\sqrt{2\gamma}) \frac{N}{\Gamma} e^{-\gamma/\Gamma} \left[1 - e^{-\gamma/\Gamma}\right]^{N-1} d\gamma. \tag{18}$$

This equation can be evaluated as a series for $N > 1$ [1]. The resulting BER has been verified by measurements in [4].

4 Maximal Ratio Combining

In the above formulation of selection diversity, we chose the element with the best SNR. This is clearly not the optimal solution as fully $(N-1)$ elements of the array are ignored. Maximal Ratio Combining (MRC) obtains the weights (see Fig. 1) that maximizes the output SNR, i.e., it is optimal in terms of SNR.

Writing the received signal at the array elements as a vector $\mathbf{x}(t)$, and the output signal as $r(t)$

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{h}(t)u(t) + \mathbf{n}(t), & (19)\\
\mathbf{h} &= \left[h_0,\, h_1,\, \ldots\, h_{N-1}\right]^T, & (20)\\
\mathbf{n} &= \left[n_0,\, n_1,\, \ldots\, n_{N-1}\right]^T, & (21)\\
r(t) &= \mathbf{w}^H\mathbf{x} = \mathbf{w}^H\mathbf{h}u(t) + \mathbf{w}^H\mathbf{n}. & (22)
\end{aligned}$$

Since the signal $u(t)$ has unit average power, the instantaneous output SNR is

$$\gamma = \frac{\left|\mathbf{w}^H\mathbf{h}\right|^2}{\mathbf{E}\left\{\left|\mathbf{w}^H\mathbf{n}\right|^2\right\}}. \tag{23}$$

The noise power in the denominator is given by

$$\begin{aligned}
P_n &= \mathrm{E}\left\{\left|\mathbf{w}^H\mathbf{n}\right|^2\right\} = \mathrm{E}\left\{\left|\mathbf{w}^H\mathbf{n}\mathbf{n}^H\mathbf{w}\right|\right\} = \mathbf{w}^H\mathbf{E}\left\{\mathbf{n}\mathbf{n}^H\right\}\mathbf{w} = \sigma^2\mathbf{w}^H\mathbf{I}_N\mathbf{w}, \\
&= \sigma^2\mathbf{w}^H\mathbf{w} = \sigma^2||\mathbf{w}||^2, & (24)
\end{aligned}$$

where $\mathbf{I}_N$ represents an $N \times N$ identity matrix. Since constants do not matter, one could always scale $\mathbf{w}$ such that $||\mathbf{w}|| = 1$. The SNR is therefore given by $\gamma = \left|\mathbf{w}^H\mathbf{h}\right|^2/\sigma^2$. By the Cauchy-Schwarz inequality, this has a maximum when $\mathbf{w}$ is linearly proportional to $\mathbf{h}$, i.e.,

$$
\begin{aligned}
\mathbf{w} &= \mathbf{h}, && (25)\\
\Rightarrow \gamma &= \frac{|\mathbf{h}^H\mathbf{h}|^2}{\sigma^2\mathbf{h}^H\mathbf{h}} = \frac{\mathbf{h}^H\mathbf{h}}{\sigma^2} = \sum_{n=0}^{N-1}\frac{|h_n|^2}{\sigma^2}, \\
&= \sum_{n=0}^{N-1}\gamma_n. && (26)
\end{aligned}
$$

The output SNR is, therefore, the sum of the SNR at each element. The best a diversity combiner can do is to choose the weights to be the fading to each element. In some sense, this answer is expected since the solution is effectively the matched filter for the fading signal. We know that the matched filter is optimal in the single user case.

Using Eqn. (26), the expected value of the output SNR is therefore N times the average SNR at each element, i.e.,

$$
\mathrm{E}\{\gamma\} = N\Gamma, \tag{27}
$$

which indicates that on average, the SNR improves by a factor of N. This is significantly better than the factor of $(\ln N)$ improvement in the selection diversity case.

To determine the pdf of the output SNR, we use fact that the pdf of the sum of N independent random variables is the convolution of the individual pdfs. Further, the convolution of two functions is equivalent to multiplying the two functions in the frequency (or Laplace) domain. We know that each γ_n in Eqn. (26) is exponentially distributed. The characteristic function of a random variable X is given by $\mathrm{E}\{e^{-sX}\}$, i.e., the characteristic function is the Laplace transform of the pdf.

$$
\begin{aligned}
F_{\Gamma_n}(s) &= \mathrm{E}\{e^{-s\gamma_n}\} = \frac{1}{1+s\Gamma}. && (28)\\
\Rightarrow F_{\Gamma}(s) &= \left[\frac{1}{1+s\Gamma}\right]^N, && (29)\\
\Rightarrow \mathrm{PDF}(\gamma) = f_{\Gamma}(\gamma) &= \mathcal{L}^{-1}\left[F_{\Gamma}(s)\right] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{e^{s\gamma}}{(1+s\Gamma)^N}d\gamma, && (30)\\
&= \frac{1}{(N-1)!}\frac{\gamma^{N-1}}{\Gamma^N}e^{-\gamma/\Gamma}, && (31)
\end{aligned}
$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

Using this pdf, the outage probability for a threshold γ_s is

$$\begin{aligned} P_{\text{out}} &= P(\gamma < \gamma_s) = \int_0^{\gamma_s} \frac{1}{(N-1)!} \frac{\gamma^{N-1}}{\Gamma^N} e^{-\gamma/\Gamma} d\gamma, \\ &= 1 - e^{-\gamma_s/\Gamma} \sum_{n=0}^{N-1} \left(\frac{\gamma_s}{\Gamma}\right)^n \frac{1}{n!}. \end{aligned} \tag{32}$$

Figure 3 illustrates the performance of a MRC with multiple antenna elements. Again, the large performance gains from using two or more elements is clear. At a outage probability of 1%, we achieve a SNR gain of greater than 8dB. To compare the performance of the MRC and selection combining techniques, we plot the outage probability for the two techniques in Fig. 4 with $N = 4$. At outage probability of 1%, MRC is about 3dB better than selection combining.

The final figure of merit is the BER in a BPSK system. The BER is given by

$$\begin{aligned} P_e &= \int_0^\infty (\text{BER}/\gamma)\, f_\Gamma(\gamma) d\gamma = \int_0^\infty \text{erfc}(\sqrt{2\gamma}) \frac{1}{(N-1)!} \frac{\gamma^{N-1}}{\Gamma^N} e^{-\gamma/\Gamma} d\gamma, \\ &= \frac{1}{(N-1)!} \left(\frac{1-\mu}{2}\right)^N \left[\sum_{n=0}^{N-1} \frac{(N-1+n)!}{n!} \left(\frac{1+\mu}{2}\right)^n\right], \qquad (33) \\ &\simeq \begin{pmatrix} 2N-1 \\ N \end{pmatrix} \left(\frac{1}{4\Gamma}\right)^N, \quad \text{for large } N, \qquad (34) \end{aligned}$$

$$\mu = \sqrt{\frac{\Gamma}{1+\Gamma}}. \tag{35}$$

The approximation for large value of N is obtained using only the final term of the summation in Eqn. (33). Note the *BER reduces exponentially as a function of N*. The rate of fall off (the exponent) is the diversity order. This is consistent with the fact that in a SISO system $P_e \propto 1/\text{SNR}$.

Definition: The a diversity system is said to have *diversity order D* if, in Rayleigh fading,

$$D = \lim_{\text{SNR}\to\infty} -\frac{\log(P_e)}{\log(\text{SNR}} \tag{36}$$

equivalently,

$$P_e \propto \frac{1}{\text{SNR}^D}, \tag{37}$$

for large SNR.

In a diversity system, therefore, we expect the BER to be a linear function of the SNR (in a log-log plot). The *slope* of the plot indicates the diversity order. The BER, in a system with diversity order two, would fall off by a factor of 10^2 for every 10dB gain in SNR.

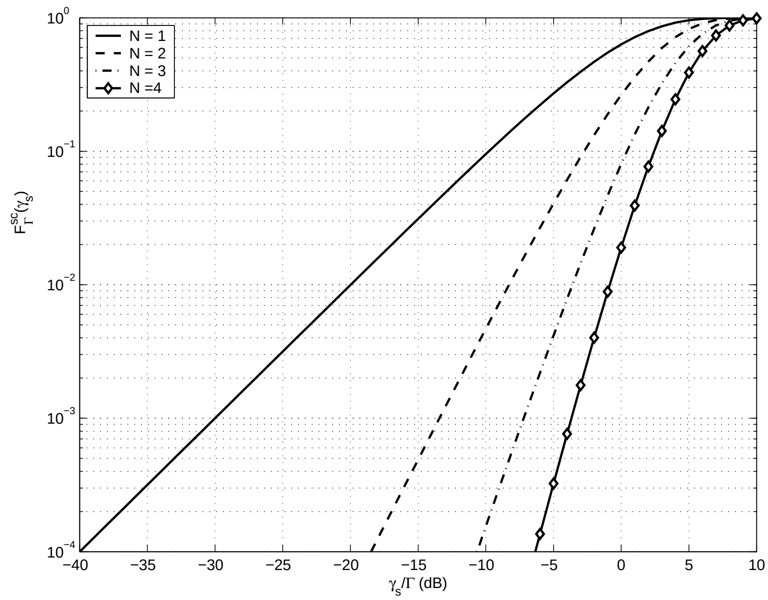

Figure 3: Performance of a maximal ratio combining system

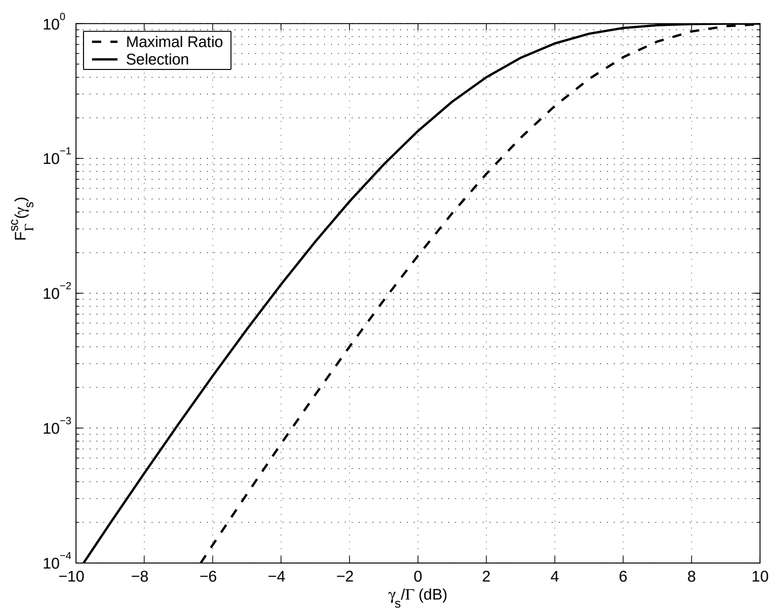

Figure 4: Comparing selection and maximal ratio combining

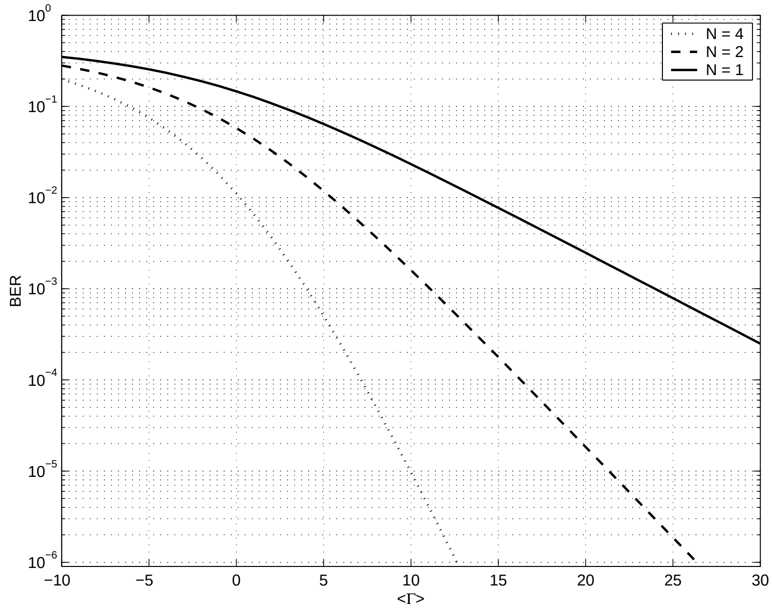

Figure 5: BER versus average SNR of a single element for Maximal Ratio Combining.

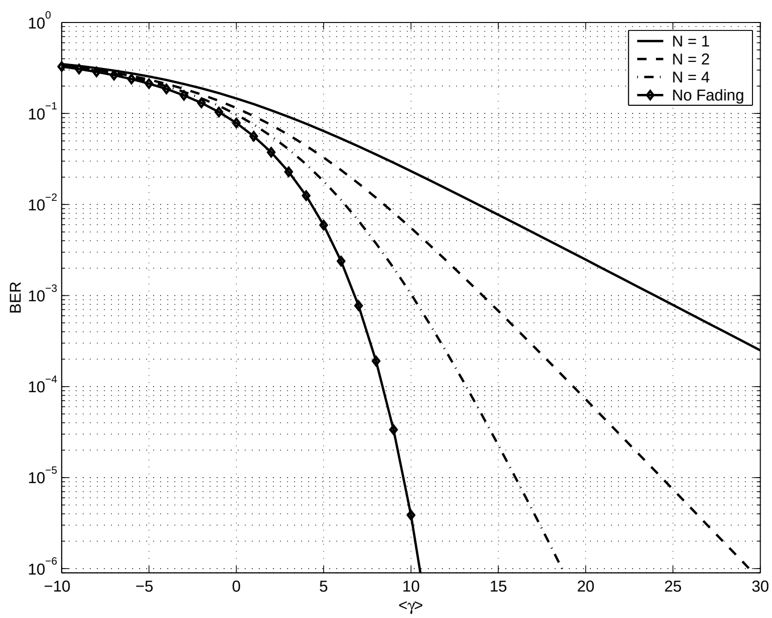

Figure 6: BER versus average output SNR for Maximal Ratio Combining.

Figure 5 plots the BER of our MRC system as a function of the average SNR at each element (Γ). The gains in performance are clear (and expected). Figure 6 plots the BER versus the output SNR ($N\Gamma$) of the MRC system. This plot proves that the MRC system *does not* act as a single element with N times the SNR. Even in this plot the slope of the BER changes as N increases. This plot is the essence of diversity - since each element receives an independently faded copy of the same signal the output SNR not only increases, but also the fluctuations in the output SNR reduce. With N copies, it is unlikely that all copies are in a deep fade, i.e., the chances of an error fall off exponentially. Note that if diversity worked just to increase output SNR (which beamforming would do), the BER would be $P_e \propto 1/(N\Gamma)$. In the diversity case, due to the fact that we are receiving independent copies, the BER falls of exponentially, $P_e \propto (1/\Gamma)^N$.

Figure 6 also illustrates the fact that line-of-sight communications is always better than communications through a fading channel - with LOS communications the BER drops off as with erfc function, i.e., exponentially in Γ. In fact, as $N \to \infty$, the MRC scheme is equivalent to LOS communications.

5 Equal Gain Combining

In Section 4 we developed the combiner that is optimal in the sense of SNR. However, the technique requires the weights to vary with the fading signals, the magnitude of which may fluctuate over several 10s of dB. The equal gain combiner sidesteps this problem by setting unit gain at each element. In the equal gain combiner,

$$w_n = e^{j\angle h_n}, \tag{38}$$

$$\Rightarrow w_n^* h_n = |h_n|, \tag{39}$$

$$\Rightarrow \mathbf{w}^H \mathbf{h} = \sum_{n=0}^{N-1} |h_n|. \tag{40}$$

Also, the noise and instantaneous SNR are given by

$$P_n = \mathbf{w}^H \mathbf{w} \sigma^2 = N\sigma^2, \tag{41}$$

$$\gamma = \frac{\left[\sum_{n=0}^{N-1} |h_n|\right]^2}{N\sigma^2}. \tag{42}$$

Using the fact that $|h_n|$ is Rayleigh distributed, using the pdf of Eqn. (4),

$$\mathrm{E}\left(|h_n|\right) = \sqrt{\pi P_0}, \tag{43}$$

$$\mathrm{E}\left(|h_n|^2\right) = P_0, \tag{44}$$

Using the SNR defined in Eqn. (42) together with Eqns. (43) and (44) we find the mean SNR is given by

$$
\begin{aligned}
\mathrm{E}\{\gamma\} &= \frac{\mathrm{E}\{\left[\sum_{n=0}^{N-1}|h_n|\right]^2\}}{2N\sigma^2} = \frac{1}{2N\sigma^2}\mathrm{E}\left\{\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}|h_n||h_m|\right\}, &(45)\\
&= \frac{1}{2N\sigma^2}\left[\mathrm{E}\left\{\sum_{n=0}^{N-1}|h_n|^2\right\} + \mathrm{E}\left\{\sum_{n=0}^{N-1}\sum_{m=0,m\neq n}^{N-1}|h_n||h_m|\right\}\right],\\
&= \frac{1}{2N\sigma^2}\left[\sum_{n=0}^{N-1}\mathrm{E}\left\{|h_n|^2\right\} + \sum_{n=0}^{N-1}\sum_{m=0,m\neq n}^{N-1}\mathrm{E}\left\{|h_n|\right\}\mathrm{E}\left\{|h_m|\right\}\right],\\
&= \frac{1}{2N\sigma^2}\left[2NP_0 + N(N-1)\sqrt{\frac{\pi P_0}{2}}\sqrt{\frac{\pi P_0}{2}}\right],\\
&= \frac{1}{2N\sigma^2}\left[2NP_0 + N(N-1)\frac{\pi P_0}{2}\right],\\
&= \left[\Gamma + (N-1)\frac{\pi\Gamma}{4}\right] = \left[1 + (N-1)\frac{\pi}{4}\right]\Gamma. &(46)
\end{aligned}
$$

The point of this analysis is to show that, despite being significantly simpler to implement, the equal gain combiner results in an improvement in SNR that is comparable to that of the optimal maximal ratio combiner. The SNR of both combiners increases *linearly* with N.

There is no closed form solution for the BER for general N, but several researchers have investigated the BER performance in several kinds of fading channels [5,6]. There are several other papers that also address this issue. In particular, in [5], Zhang finds the closed form solutions in Rayleigh fading for $N = 2$ and $N = 3$ based on the characteristic function method. For BPSK they are:

$$P_e = \frac{1}{2}\left[1 - \frac{\sqrt{\Gamma(\Gamma+2)}}{\Gamma+1}\right], \quad N = 2, \tag{47}$$

$$P_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\Gamma(2\Gamma+3)^2}{3(\Gamma+1)^3}} \times {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \frac{\Gamma^2}{(2\Gamma+3)^2}\right) + \frac{\pi}{4}\sqrt{\frac{\Gamma^3}{27(\Gamma+1)^3}}, \quad N = 3, \tag{48}$$

where ${}_2F_1$ is a hypergeometric function. See [6] for more general results.

Figure 7 plots the BER for the case of equal gain combining. This plot is taken from Zhang [5]. Again the significant gains in performance are clear. In the plot the number of elements (our N) is represented as L. Note again that an array with N elements provides order-N diversity.

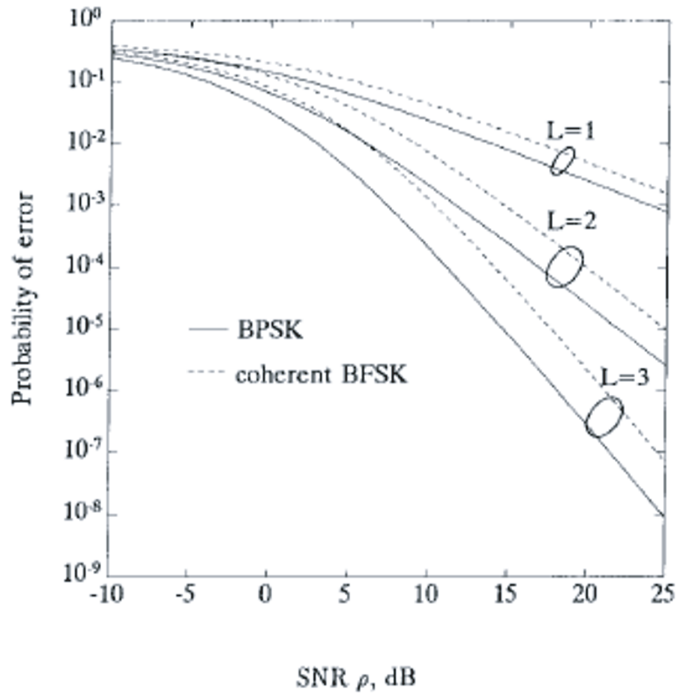

Figure 7: BER versus SNR of a single element

6 Comparison of the Three Techniques

We compare the performance of the three techniques in terms of the complexity and improvement in SNR. Figure 8 plots the improvement in SNR as a function of the number of elements. As expected the best improvement is for the maximal ratio combiner, while the worst is for the selection diversity technique. Note that the improvement in the case of equal gain combining is comparable to that of maximal ratio combining.

In terms of the required processing, the selection combiner is the easiest - it requires only a measurement of SNR at each element, not the phase or the amplitude, i.e., this combiner need not be coherent. Note, however, that the results presented use a coherent receiver (the phase of channel is removed after the fact). Both the maximal ratio and equal gain combiners, on the other hand, require phase information. The maximal ratio combiner requires accurate measurement of the gain too. This is clearly difficult to implement, as the dynamic range of a Rayleigh fading signal may be quite large. For this additional cost, for two elements, the MRC improves performance by about 0.6dB over the equal gain combiner at a BER of 1%.

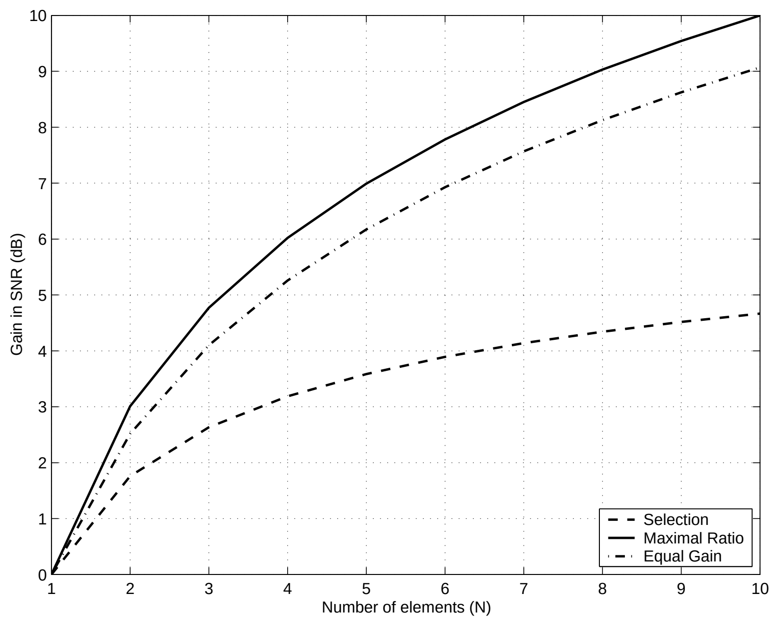

Figure 8: Improvement in SNR for the three kinds of combining

7 Diversity Order, Diversity Gain versus Antenna Gain

Diversity *order* is defined as the *slope of the BER curve* as a function of SNR (in a log-log scale), i.e., for diversity order D, BER $\propto$ (SNR)$^{-D}$. Note that this slope is the same for the outage probability too. The key to diversity is that each element is assumed to receive an *independent* copy of the same signal, overall there are N independent copies. The performance gains arise because it is unlikely that all these independent samples of the fading process are in a deep fade.

Diversity order measures how many "independent" copies are available. For the analysis so far all N elements receive independent copies and so the diversity order for all cases is N. However, as we shall while discussing more realistic scenarios, any correlation between elements reduces this measure of independence. Diversity order, therefore, tells us how much independence is "left over". For example, if all the channels were perfectly correlated, if one channel were in a deep fade, all channels would be in a deep fade, i.e., there is no diversity at all. In addition to the analysis of receive diversity here, diversity order will play an important role when we discuss transmit diversity, especially space-time coding.

Diversity gain is defined as the improvement in some resource (usually power) required to achieve a certain performance criterion (usually a required BER). Hence, from Fig. 3, the diversity gain for MRC is approximately 13dB for $N = 2$. Note that there exists a law of diminishing marginal returns in diversity gain. The additional gain by adding another element is only about 5dB (the diversity gain for $N = 3$ is approximately 18dB).

Antenna gain, on the other hand, is the gain in output signal-to-noise ratio, i.e., antenna gain tells us how the use of multiple elements improves signal strength. As shown before, the antenna gain for MRC is N while for selection combining its of order of $\ln(N)$.

8 Impact of Correlation

So far we have assumed that the fading signal at each element is independent of the signal at other elements. This is the ideal situation, one that is not achievable in practice. (This is one of the few cases where signal processing researchers can completely ignore electromagnetism and still arrive at a non-ideal situation!). In this section we will analyze the causes, magnitude and impact of correlation between the signals at two elements. Note that we will only address cross-correlation (uncorrelated signals need not be independent). However, we know that for Gaussian fading (Rayleigh fading is complex Gaussian) uncorrelated fading also implies independent fading.

There are two main causes of correlation between signals. The first, from the point of view of electromagnetism, is due to the mutual coupling between elements. The second, is the fact that even after propagating through the channel, the signals at two elements of the receiving array are not completely decorrelated. In point of fact, the correlation is dependent on the spacing between the elements - in general, increased spacing implies decreased correlation.

In an array of two elements, using MRC, if the signals at the two elements are individually Rayleigh distributed and have a correlation of ρ, the pdf of the SNR, outage probability, and BER are given by [1]:

$$f_\Gamma(\gamma) = \frac{1}{2|\rho|\Gamma}\left[e^{-\gamma/(1+|\rho|\Gamma)} - e^{-\gamma/(1-|\rho|\Gamma)}\right] \tag{49}$$

$$P_{\text{out}}(\gamma_s) = 1 - \frac{1}{2|\rho|}\left[(1+|\rho|)e^{-\gamma_s/(1+|\rho|\Gamma)} - (1-|\rho|)e^{-\gamma_s/(1-|\rho|\Gamma)}\right] \tag{50}$$

$$P_e = \frac{1}{4|\rho|}\left[(1+|\rho|)\left(1 - \frac{1}{1+\frac{1}{(1+|\rho|\Gamma)}}\right) - (1-|\rho|)\left(1 - \frac{1}{1+\frac{1}{(1-|\rho|\Gamma)}}\right)\right] \tag{51}$$

Figure 9 plots the outage probability of an MRC system for various values of correlation. As is clear, going from perfect decorrelation (as assumed in section 4) to perfect correlation ($\rho = 1$) results in significant loss of performance. At about a 1% outage probability, the loss is about 8dB. In the rest of this section we will investigate models to determine the correlation. Note that for a correlation below $\rho = 0.5$ the performance degradation is negligible. In practice, researchers assume uncorrelated fading if this criterion is met (analyzing independent fading is much easier than correlated fading). In designing a system to exploit spatial diversity one would generally space the antenna elements to ensure $\rho < 0.5$.

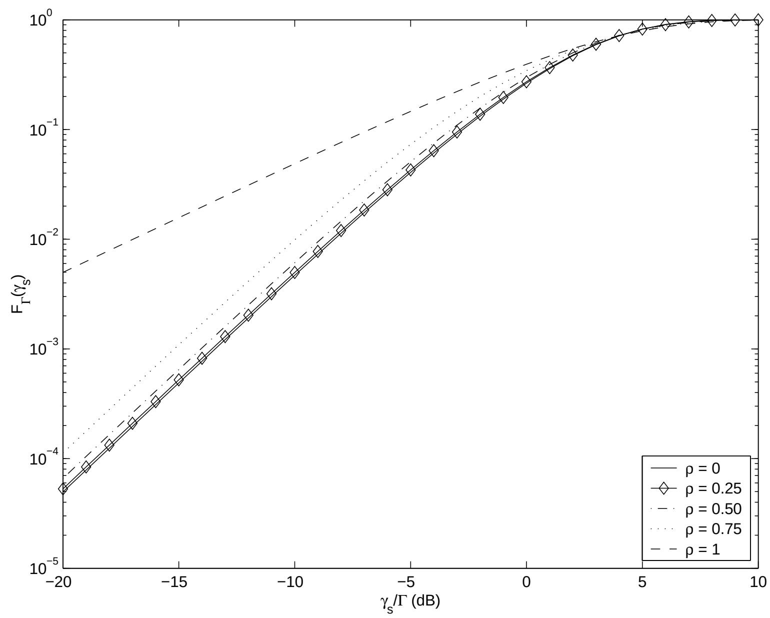

Figure 9: Probability of outage for changing correlation

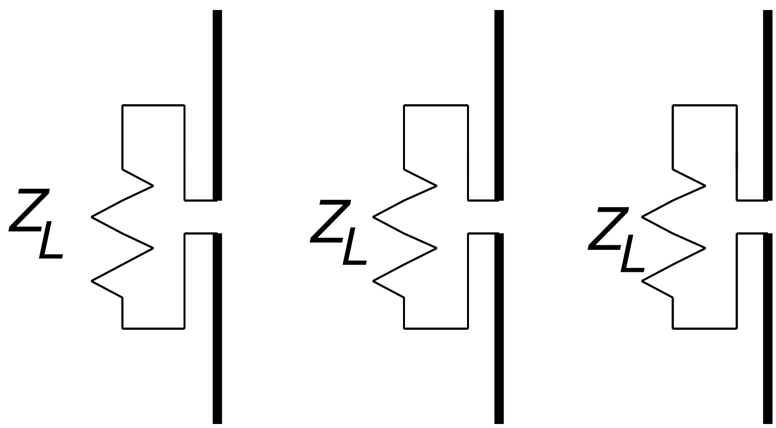

Figure 10: An array of dipoles

8.1 Correlation due to mutual coupling

Consider an array of N elements, an illustration of which is given in Fig. 10. The elements here are dipoles, but the following argument is valid in general. The incoming signal causes a current in each element. The current radiates causing a reaction signal in the other elements, i.e., *all elements are mutually coupled*, clearly voiding the assumption of independence.

Compensation for the mutual coupling starts by recognizing that eliminating the currents on the array would also eliminate the mutual coupling. The simplest formulation for compensation suggests that if the current at the port were set to zero (the port were open-circuited), the mutual coupling would be eliminated. In practice, there is a residual mutual coupling due to the fact that there is a residual current on the arms of the dipole. However, that is a second order effect which may prove to be important in a realistic system. Analysis of the second order effect is given in [7,8].

As we are attempting to (virtually) open circuit the dipoles, we are effectively focusing on the ports of the array - we will analyze the array as a N-port network. If the signal received at the n-th element is V_n and the open circuit voltage at the same element is $V_{\mathrm{oc}n}$. The resulting equations can be written as

$$V_n = Z_L I_n \tag{52}$$

$$\Rightarrow \mathbf{V} = \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{N-1} \end{bmatrix} = \begin{bmatrix} Z_L & 0 & 0 & \cdots 0 & 0 \\ 0 & Z_L & 0 & \cdots 0 & 0 \\ 0 & 0 & Z_L & \cdots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots Z_L & 0 \\ 0 & 0 & 0 & \cdots 0 & Z_L \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ \vdots \\ I_{N-1} \end{bmatrix} \tag{53}$$

$$= \mathbf{Z}_L \mathbf{I} \tag{54}$$

But, treating the array as an N port network, results in the impedance matrix formulation

$$\begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_{N-1} \end{bmatrix} = \begin{bmatrix} Z_{00} & Z_{01} & Z_{02} & \cdots Z_{0(N-2)} & Z_{0(N-1)} \\ Z_{10} & Z_{11} & Z_{12} & \cdots Z_{1(N-2)} & Z_{1(N-1)} \\ Z_{20} & Z_{21} & Z_{22} & \cdots Z_{2(N-2)} & Z_{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_{(N-2)0} & Z_{(N-2)1} & Z_{(N-2)2} & \cdots Z_{(N-2)(N-2)} & Z_{(N-2)(N-1)} \\ Z_{(N-1)0} & Z_{(N-1)1} & Z_{(N-1)2} & \cdots Z_{(N-1)(N-2)} & Z_{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} I_0 \\ I_1 \\ \vdots \\ I_{N-1} \end{bmatrix} + \begin{bmatrix} V_{\mathrm{oc}0} \\ V_{\mathrm{oc}1} \\ \vdots \\ V_{\mathrm{oc}(N-1)} \end{bmatrix} \tag{55}$$

$$\mathbf{V} = \mathbf{Z}\mathbf{I} + \mathbf{V}_{\mathrm{oc}} \tag{56}$$

Using Eqns. (54) and (56), we get

$$\mathbf{V}_{\mathrm{oc}} = \left[\mathbf{Z} + \mathbf{Z}_L\right] \mathbf{Z}_L^{-1} \mathbf{V}, \tag{57}$$

i.e., it is possible to evaluate the open circuit voltages (the voltages compensated for mutual coupling) using the measured voltages and knowledge of the port impedance matrix. Similarly, Eqn. (57) can be used to model mutually coupled signals. The vector $\mathbf{V}_{\mathrm{oc}}$ is the signal without mutual coupling, while the vector $\mathbf{V}$ is the mutually coupled received signal.

The port impedance matrix can be obtained using EM analysis tools such as the Method of Moments [9]. It is instructive to think of the open circuit voltages and the impedances as the Thevenin equivalent of the port.

The goal of the network analysis above is to compensate for the mutual coupling, i.e., it is hoped that the correlation between the signals after compensation (the open circuit signals) is lower than before compensation (the measured signals in vector $\mathbf{V}$). Note that this compensation does not account for the fact that the incoming signals may themselves be correlated from one element to the next. This is the subject of the discussion in the next section.

8.2 Correlation modelling a realistic channel

In developing the theory of diversity combining, we assumed the channel to be so severe that the short difference in the path from the transmitter and one element and the path from the transmitter and another element in the array is "enough" to cause the received signals at the two elements to be independent. In a more realistic channel, this is not the case and there is a residual correlation between the received signals. The degree of correlation depends on the channel and the physical location of the receiver antenna with respect to its surrounding scatterers. We will therefore consider two different cases: diversity at the mobile (antenna surrounded by scatterers in all directions) and diversity at the base station (assumed to be at a sufficient height that the received signals arrive from a limited angular region).

Consider an array placed along the x-axis and an incoming signal from angle (θ, ϕ). Using the steering vector derived earlier the signals at the two elements are $s_0 = 1$ and $s_1 = e^{jkdcos\phi\sin\theta}$, where d is the spacing between the two elements. Remember, in class, we ignored the θ term. The correlation is defined as

$$\begin{aligned}
\rho &= \frac{\mathrm{E}\{s_0 s_1^*\}}{\sqrt{\mathrm{E}\{|s_0|^2\}\mathrm{E}\{|s_1|^2\}}}, \\
&= \frac{\mathrm{E}\{s_0 s_1^*\}}{\mathrm{E}\{|s_0|^2\}}, \qquad (58)
\end{aligned}$$

$$\mathrm{E}\{|s_0|^2\} = \int_0^\pi \int_0^{2\pi} f_{\Theta,\Phi}(\theta,\phi) d\phi\, d\theta, \qquad (59)$$

$$\mathrm{E}\{s_1 s_0^*\} = \int_0^\pi \int_0^{2\pi} e^{jkdcos\phi\sin\theta} f_{\Theta,\Phi}(\theta,\phi) d\phi\, d\theta, \qquad (60)$$

where $f_{\Theta,\Phi}(\theta,\phi)$ is the pdf of the power in the incoming signal as a function of the elevation and azimuthal angles. In practice, this pdf would be either be replaced with a measurement of the power received as a function of the two angles *or a realistic model would be chosen and the parameters of*

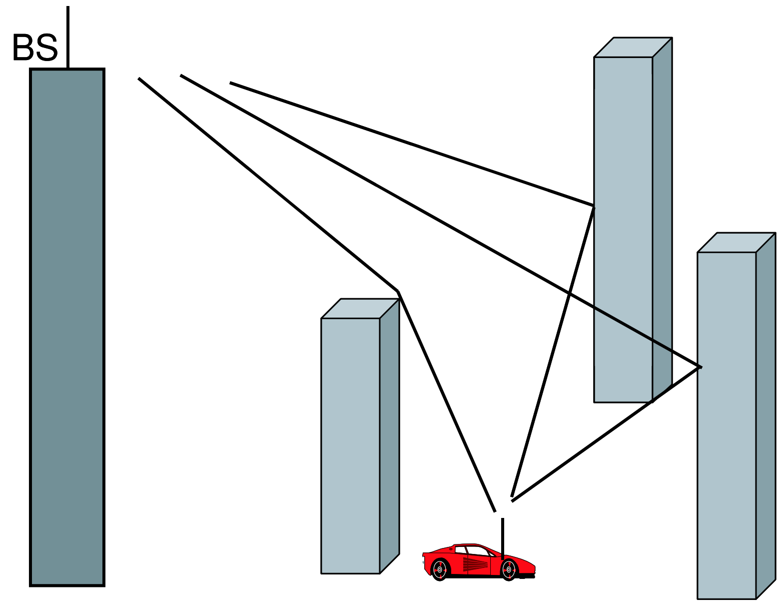

Figure 11: The model for angle spread in a communication system

the model would be fit to the available measurements. A common choice of parameter to match is the RMS angle spread.

Equations (58-60) are the general form of the correlation between the signals at two elements placed along the x-axis. As we have done throughout, we will choose particular (simple) models for each term to get a feel for the degree of correlation. Here we will assume that the overall power pdf can be written as the product of two marginal pdfs, i.e., $f_{\Theta,\Phi}(\theta, \phi) = f_{\Theta}(\theta) f_{\Phi}(\phi)$.

We now distinguish between correlation at the mobile and correlation at the base station. The physical model we consider is shown in Fig. 11. The mobile is at ground height, surrounded by scatterers in all directions. The basestation, on the other hand, is at a considerable height over the surrounding scatterers. This simple model is valid for macrocell environments. Note that other models have been developed for other environments [1,10]. This model for the physical surroundings of the receiving antenna will play a significant role in the correlation between diversity branches. Realistic models for the resulting power pdfs in these two cases will result in hugely different results.

8.2.1 Spatial correlation at the mobile

At the mobile, as the receiving antenna is surrounded by scatterers, on average, the received power is uniformly distributed over the azimuthal angle, i.e.,

$$f_\Phi(\phi) = \frac{1}{2\pi}, \quad 0 < \phi < 2\pi, \tag{61}$$

$$\Rightarrow \rho = \frac{1}{2\pi}\int_0^\pi \int_0^{2\pi} e^{-jkd\cos\phi\sin\theta} f_\Theta(\theta) d\phi\, d\theta, \tag{62}$$

$$= \int_0^\pi J_0(kd\sin\theta) f_\Theta(\theta) d\theta, \tag{63}$$

where $J_0(x)$ represents the zero$^{\text{th}}$ order Bessel function of the first kind, defined as

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos[x\cos\phi] d\phi$$

Note that the imaginary part of integral in Eqn. (62) is zero. For some sample realistic pdf choices for Θ, we can now evaluate the correlation between the signals at two elements [1].

$$f_\Theta(\theta) = \delta(\theta - \theta_0) \Rightarrow \rho = J_0(kd\sin\theta_0) \tag{64}$$

$$f_\Theta(\theta) = ae^{-b\cos\theta} \Rightarrow \rho = \frac{b}{e^b - 1}\frac{kd\sin(kd) - b\cos(kd) + be^b}{b^2 + (kd)^2} \tag{65}$$

The first model assumes that all the energy from the basestation arrives from the same elevation angle, i.e., elevation angle spread may be ignored. The second (and more realistic) model implies that most of the energy arrives from broadside (elevation angle of $\pi/2$), but the power drops off exponentially. In practice, one would measure the received power (as a function of angle) at various mobile locations and match the resulting RMS spread to the model above (to determine the parameter b. The parameter a acts as a normalization and is determined from b) .

For the δ-function model, we see that even with complete loss of direction information (the signals arrive with equal power/probability from all angles) the resulting correlation is non-zero. In fact, the correlation falls off as the zero$^{\text{th}}$ order Bessel function. Note that the Bessel function is *not* uniformly decreasing. Choosing a maximum allowed correlation sets the minimum spacing between elements.

Using the δ-function model, if we want a correlation below 0.5, we need $kd\sin\theta_0 > 1.52$ and for $\theta_0 = \pi/2$, $d > 0.24\lambda$. Any other value of θ_0 will only increase the required spacing between elements. In a realistic system, therefore, we need a spacing on order of half a wavelength for reasonable diversity gain.

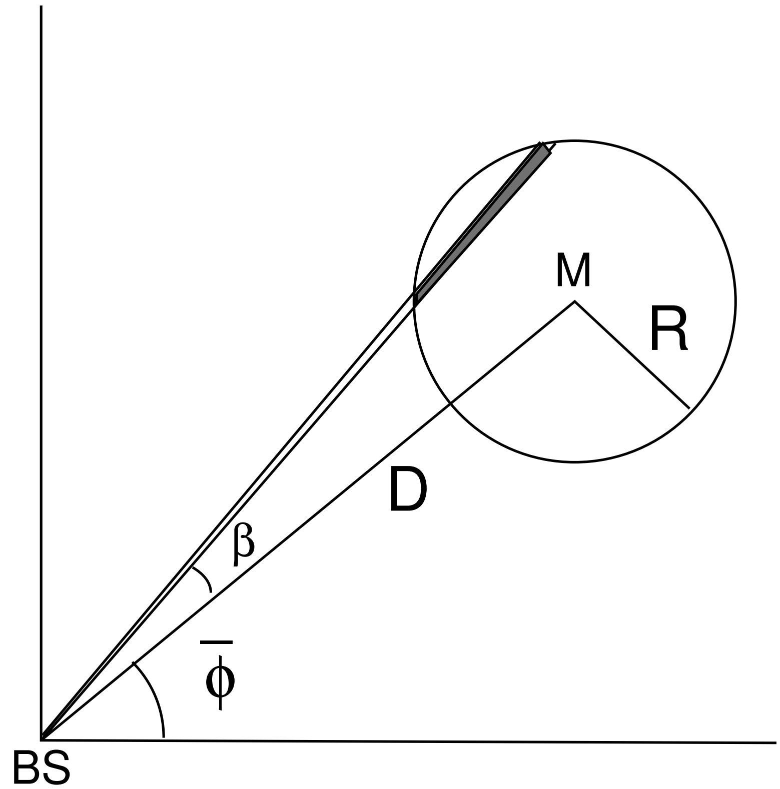

Figure 12: A disk of scatterers surrounding the mobile.

8.2.2 Spatial correlation at the basestation

The situation for the basestation is considerably more complicated. The model shown in Fig. 11 implies that the signals arrive at the basestation from a relatively narrow azimuthal region, after reflecting and refracting off *local scatterers near the mobile*. The model we will use then is to assume the signals arrive in elevation from a single angle ($f_\Theta(\theta) = \delta(\theta - \theta_0)$). In azimuth, the signals arrive from within a disk of scatterers surrounding the mobile. On average the transmit power is distributed uniformly throughout the disk.

Figure 12 shows a top view of the physical situation. The mobile (marked M on the figure) is at the center of a disk of scatterers of radius R, a distance D from the basestation (marked BS). Due to scattering, the power transmitted by the mobile is uniformly distributed within this disk. The basestation is at the origin. Remember that the array is assumed oriented along the x-axis. The azimuthal angle to the center of the disk is marked as $\bar{\phi}$, and the incremental angle β, i.e., any point within the disk is at azimuthal angle $\phi = \bar{\phi} + \beta$. Assuming the power distribution to be uniform within this disk, we need to translate this power distribution to a pdf for the azimuthal angle ϕ (or equivalently β). By definition of the pdf,

$$\begin{aligned}
f_B(\beta)d\beta &= \text{Power received from within angle } \beta \text{ and } (\beta + d\beta), \\
&= \frac{1}{\pi R^2} \times (\text{Area of shaded region}), \\
&= \frac{1}{\pi R^2} \times \left[\frac{r_M + r_m}{2}\, d\beta\right] \times (r_M - r_m), \\
&= \frac{1}{\pi R^2} \times \frac{r_M^2 - r_m^2}{2} d\beta, \\
\Rightarrow f_B(\beta) &= \frac{1}{\pi R^2}\frac{r_M^2 - r_m^2}{2}, \qquad (66)
\end{aligned}$$

where r_M is the distance to point of intersection of the line at angle β and the farther point on the circle, while r_m is the distance to the near point on the circle. Therefore,

$$\begin{aligned}
r_m &= D\cos\beta - \sqrt{R^2 - D^2\sin^2\beta}, \\
r_M &= D\cos\beta + \sqrt{R^2 - D^2\sin^2\beta}, \\
\Rightarrow r_M^2 - r_m^2 &= 4D\cos\beta\sqrt{R^2 - D^2\sin^2\beta}.
\end{aligned}$$

Recognizing that the range of β is fixed by $\sin\beta_{\max} = R/D$, the pdf for β is given by

$$f_B(\beta) = \frac{2\cos\beta\sqrt{\sin^2\beta_{\max} - \sin^2\beta}}{\pi\sin^2\beta_{\max}} \quad |\beta| < \beta_{\max}. \qquad (67)$$

Note that this pdf is valid only for the uniform disk of scatterers model assumed. Other models are discussed in [1, 10].

Based on this pdf, the resulting correlation is given by

$$\rho = \int_{-\beta_{\max}}^{\beta_{\max}} e^{-jkd\cos(\bar{\phi}+\beta)\sin\theta_0} f_B(\beta)d\beta, \qquad (68)$$

which can only be evaluated in closed form for $\bar{\phi} = \pi/2$. Note that since the array is assumed to be oriented along the x-axis, we cannot pick the axes to have $\bar{\phi} = \pi/2$ *always*. For this case,

$$\rho = \frac{2J_1(kd\sin\beta_{\max}\sin\theta_0)}{kd\sin\beta_{\max}\sin\theta_0}. \qquad (69)$$

For $\rho < 0.5$, we get $kd\sin\beta_{\max}\sin\theta_0 > 2.026$. Choosing sample values of $R = 1.2$km, $D = 50$m, $\theta_0 = 80^o$ we get $d > 8.8\lambda$.

There are two important issues to note here. As always, the analysis and numbers presented here are ball park figures only. This analysis may not be valid if the underlying model is not valid. Due to the fact that the signals arrive from within a range of $\beta_{\max}$, the correlation between elements increases significantly. However, since the base station is expected to have more available "real estate", this increased spacing may not pose a significant problem. As the mobile moves further and further away, $\beta_{\max}$ decreases, resulting in *increased correlation*. This is expected from a physical point of view, as the signals appear to arrive from a narrower spread. Remember that if the signal arrived from a single point, the signals at any two elements would be perfectly correlated.

9 Interference Cancellation Coupled with Diversity

The analysis, so far, has focused exclusively on a single user situation, assuming no interference. However, clearly, this is not a practical situation. In discussing beamforming we saw several iterative algorithms to suppress interference - these algorithms required the interference statistics to remain constant while converging to the optimal weights. This section may also be considered an analysis of the performance of interference cancellation, something we did not attempt in the previous chapter on interference cancellation. In practice this meant the channel vectors remain constant while the iterative process converges. The signal model is

$$\begin{aligned}\mathbf{x} &= \mathbf{h}_0 + \sum_{m=1}^{M} \mathbf{h}_m + \mathbf{n}, &(70)\\ &= \mathbf{h}_0 + \mathbf{x}_i, &(71)\end{aligned}$$

where $\mathbf{h}_m$ is the channel vector associated with the m-th interfering user and $\mathbf{x}_i$ is the vector of interference-plus-noise. Due to the Doppler frequency of a moving user, each of these vectors is changing with time. Therefore, interference cancellation only makes sense if these channels are approximately constant over some significant length of time.

Using back-of-the-envelope calculations, for a user moving at $v = 30m/s$ (108kmph) and a center frequency of $f_0 = 1$GHz ($\lambda_0 = 0.3m$) the Doppler frequency is $f_d = v/\lambda_0 = 30/0.3 = 100$Hz. The associated channel, therefore, changes at a rate of approximately 100 times a second. Assuming a relatively low data rate of 10ksps, the channel remains approximately constant for 100 symbols, i.e., interference suppression *is possible*, though one should plan on using a scheme that converges to the required solution well within these 100 symbols.

Having shown that interference suppression is possible, the analysis below assumes we can converge to the optimal weights within the available time. Defining the interference covariance matrix to be $\mathbf{R}_I = \mathrm{E}\left[\mathbf{x}_i\mathbf{x}_i^H\right]$, the optimal weights are $\mathbf{w} = \mathbf{R}_I^{-1}\mathbf{h}_0$. Since the process must converge within a short time, this is a *short-time covariance matrix*. Note that

$$\mathbf{R}_I = \sum_{m=1}^{M} \mathbf{h}_m \mathbf{h}_m^H + \sigma^2 \mathbf{I}, \tag{72}$$

since $\mathrm{E}\left[|\alpha_m|^2\right] = 1$. The matrix itself will change over larger time scales (time scales associated with the Doppler rate). The *instantaneous* output signal-to-interference-plus-noise ration (SINR) is given by

$$\begin{aligned} \gamma &= \frac{\left|\mathbf{w}^H \mathbf{h}_0\right|^2}{\mathrm{E}\left[\mathbf{w}^H \mathbf{x}_i \mathbf{x}_i \mathbf{w}\right]} = \frac{\left|\mathbf{w}^H \mathbf{h}_0\right|^2}{\mathbf{w}^H \mathbf{R}_I \mathbf{w}} = \frac{\left|\mathbf{h}_0^H \mathbf{R}_I^{-1} \mathbf{h}_0\right|^2}{\mathbf{h}_0^H \mathbf{R}_I^{-1} \mathbf{R}_I \mathbf{R}_I^{-1} \mathbf{h}_0}, \\ &= \mathbf{h}_0^H \mathbf{R}_I^{-1} \mathbf{h}_0. \end{aligned} \tag{73}$$

The SNR is dependent on two factors that change randomly over large time scales, $\mathbf{R}_I$ and $\mathbf{h}_0$. Finding the average SNR requires averaging over these two random processes. Denoting as $\langle \cdot \rangle$ the expectation over these large time scales, $\langle \cdot \rangle_0$ as this expectation over $\mathbf{h}_0$ and $\langle \cdot \rangle_I$ as the expectation over $\mathbf{R}_I$,

$$\langle \gamma \rangle = \left\langle \left\langle \mathbf{h}_0^H \mathbf{R}_I^{-1} \mathbf{h}_0 \right\rangle_0 \right\rangle_I. \tag{74}$$

Denoting as $\Psi(s)$ the characteristic function of γ,

$$\Psi(s) = \left\langle \left\langle e^{-s\gamma} \right\rangle_0 \right\rangle_I, \tag{75}$$

Using the fact that $\mathbf{h}_0$ is complex normal, define $\mathbf{Q}_0 = \mathrm{E}\left[\mathbf{h}_0 \mathbf{h}_0^H\right]$ as the long-term covariance matrix of the fading from the desired user, i.e., $\mathbf{h}_0 \sim \mathcal{CN}(\mathbf{0}, \mathbf{Q}_0)$.

$$\begin{aligned} \left\langle e^{-s\gamma} \right\rangle_0 &= \frac{1}{(\pi^N \det \mathbf{Q}_0)} \int_{\mathbf{h}_0} \exp\left[-\mathbf{h}_0^H \mathbf{Q}_0^{-1} \mathbf{h}_0 - s\gamma\right] d\mathbf{h}_0, \\ &= \frac{1}{(\pi^N \det \mathbf{Q}_0)} \int_{\mathbf{h}_0} \exp\left[-\mathbf{h}_0^H \left(\mathbf{Q}_0^{-1} + s\mathbf{R}_I^{-1}\right) \mathbf{h}_0\right] d\mathbf{h}_0, \end{aligned} \tag{76}$$

where the integrals are all N-fold. Using the fact that any pdf must integrate to 1, the final integral is equal to

$$\begin{aligned} \int_{\mathbf{h}_0} \exp\left[-\mathbf{h}_0^H \left(\mathbf{Q}_0^{-1} + s\mathbf{R}_I^{-1}\right) \mathbf{h}_0\right] d\mathbf{h}_0 &= \pi^N \det\left(\mathbf{Q}_0^{-1} + s\mathbf{R}_I^{-1}\right)^{-1}, \\ \Rightarrow \left\langle e^{-s\gamma} \right\rangle_0 &= \frac{1}{(\pi^N \det \mathbf{Q}_0)} \pi^N \det\left(\mathbf{Q}_0^{-1} + s\mathbf{R}_I^{-1}\right)^{-1} \\ &= \frac{1}{\det\left(\mathbf{I}_N + s\mathbf{Q}_0 \mathbf{R}_I^{-1}\right)}. \end{aligned} \tag{77}$$

Therefore,

$$\Psi(s) = \left\langle \frac{1}{\det\left(\mathbf{I}_N + s\mathbf{Q}_0 \mathbf{R}_I^{-1}\right)} \right\rangle_I, \tag{78}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix.

Denoting as $\{\lambda_n,\, n = 1, \ldots, N\}$ the eigenvalues of $\mathbf{R}_I \mathbf{Q}_0^{-1}$, Eqn. (78) can be re-written as

$$\begin{aligned} \Psi(s) &= \left\langle \prod_{n=1}^{N} \frac{1}{1 + \frac{s}{\lambda_n}} \right\rangle_I, & (79) \\ &\simeq \prod_{n=1}^{N} \frac{1}{1 + \frac{s}{\langle \lambda_n \rangle_I}}. & (80) \end{aligned}$$

The final approximation, developed in [11], is a significant simplification of the analysis.

Finding the average SINR ($\langle \gamma \rangle$) requires a partial fraction expansion of this result. Under independent fading, if there are no interfering users, clearly $\mathbf{R}_I = \sigma^2 \mathbf{I}_N$ and $\mathbf{R}_0 = |\alpha_0|^2 \mathbf{I}_N$. The eigenvalues of $\mathbf{R}_I \mathbf{R}_0^{-1}$ are $\lambda_n = \sigma^2 / |\alpha_0|^2$, i.e., all the eigenvalues are equal to the inverse of the average SNR. For each interfering signal, the one eigenvalue changes. Therefore, the number of unique (non-repeated) eigenvalues are the number of interfering users (M). Assuming $M < N$, the partial fraction expansion is therefore given by

$$\Psi(s) = \sum_{n=1}^{M} \frac{B_n}{s + \langle \lambda_n \rangle_I} + \sum_{n=1}^{N-M} \frac{C_n}{(s + \langle \lambda_N \rangle_I)^n}. \tag{81}$$

The pdf of the SINR is the inverse Laplace transform of this characteristic function.

$$\begin{aligned} \Rightarrow f_\Gamma(\gamma) &= \sum_{n=1}^{M} B_n e^{-\gamma \langle \lambda_n \rangle_I} + e^{-\gamma \langle \lambda_N \rangle_I} \sum_{n=1}^{N-M} C_n \frac{\gamma^{n-1}}{(n-1)!} & (82) \\ \langle \gamma \rangle &= \sum_{n=1}^{N} \frac{1}{\langle \lambda_n \rangle_I} & (83) \end{aligned}$$

The crucial fact to recognize is that the second term in the pdf of Eqn. (82) is exactly the same as for the MRC. However, note that the diversity order is $(N - M)$. Note that if $M = 0$, the result collapses to the MRC case. With N elements in the array, there are N degrees of freedom. M of these degrees of freedom are used to suppress interferers, leaving only $(N - M)$ left over for diversity gains. Therefore, there is a trade off between interference suppression and diversity gains. There is no free lunch!!

10 Summary

In this discussion of diversity, we have investigated Selection (the easiest and least optimal), Maximal Ratio (the optimal and most difficult) and Equal Gain (a trade off between the two). Selection combining required no phase shifters or gain elements, only the measurement of SNR at each element. However, the gain in output SNR rose only on order of $\ln N$. MRC required both phase shifters (a coherent receiver) and variable gains at each element. The gain in SNR rose as N, the best possible scenario. Finally, EGC required only phase shifters while still providing gain in SNR on order of N. In the case of uncorrelated fading, all three schemes provide diversity order of N.

We also discussed the impact of correlation between branches and developed models for the correlation at the mobile and the basestation separately. We then obtained *ball park* figures for required spacing between elements for "adequate" decorrelation.

Finally, we investigated simultaneous interference suppression and diversity combining. The suppression of each interfering source reduces the diversity order by one. Optimal interference cancellation, therefore, results in a *trade-off* between cancellation and diversity.

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