Improved Sum-Rate Optimization in the Multiuser MIMO Downlink

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***Abstract*—We consider linear precoding and decoding in the downlink of a multiuser multiple-input, multiple-output (MIMO) system. In this scenario, the transmitter and the receivers may each be equipped with multiple antennas, and each user may receive more than one data stream. We examine the relationship between the sum capacity for the broadcast channel with channel state information at the transmitter under a sum power constraint and the achievable sum rates under linear precoding. We show that achieving the optimum sum throughput under linear precoding is equivalent to minimizing the product of mean squared error (MSE) matrix determinants. The resulting non-convex optimization problem is solved numerically, guaranteeing local convergence only. The performance of this approach is analyzed via comparison to the sum capacity and to existing approaches for linear precoding.**

I. Introduction

A relatively recent theme in multiple-input multiple-output (MIMO) research involves its application to the *multiuser* downlink, where a single base station communicates with multiple users on the same time/frequency channel. MIMO techniques enable improved reliability and/or increased data rates by exploiting the spatial dimension using an antenna array at the transmitter and/or the receiver. In this paper, we focus on using these methods to maximize total throughput.

Communication in the multiuser MIMO downlink, also known as the broadcast channel (BC), has been of particular interest in information theory. The sum capacity in the BC has been characterized using an uplink-downlink duality [1], [2] and game theory [3], and optimum strategies that achieve sum capacity [4], [5] have been derived using Costa's dirty paper coding (DPC) [6]. Practical realizations of DPC are largely based on Tomlinson-Harashima precoding (THP) [7]–[10]. These methods incur high complexity due to their nonlinear nature and the combinatorially complex problem of finding an optimal user ordering. THP-based schemes also suffer from rate loss when compared to the sum capacity due to modulo and shaping losses.

Several researchers have considered linear precoding as an alternative to THP-based approaches to reduce precoder complexity in the MIMO downlink. The analysis in [11] compares zero forcing (ZF) and its generalization, block diagonalization (BD), in the asymptotically high SNR region. Both of these techniques use simple linear precoding techniques to transform the multiuser channel into orthogonal single-user channels; however, the decreased complexity comes at the expense of restricting the number of receive antennas to be fewer than the number of transmit antennas. These schemes, therefore, limit the possibility of gains from additional receiver antennas.

Linear precoding approaches to sum rate maximization with higher complexity have been proposed for single-antenna receivers [12], [13] and for multiple antenna receivers [14], [15]. In [12], the authors suggest an iterative method for direct optimization of the sum rate, while [13] and [14] exploit the SINR uplink-downlink duality of [16]–[18]. In [15] and [19], two similar algorithms were independently proposed to maximize the sum rate indirectly, by formulating the problem as the minimization of the product of the mean squared errors (PMSE). The work of [19] was motivated by the the equivalence relationship developed between the single user minimum MSE (MMSE) and mutual information in [20]. Each of these approaches in [12]–[15], [19] yields a suboptimal solution, as the resulting solutions converge only to a local optimum, if at all.

In the single-user multicarrier case, minimizing the PMSE is equivalent to minimizing the determinant of the MSE matrix and thus is also equivalent to maximizing the mutual information [21]. This equivalence does not exist in the single-carrier multiuser downlink. In this paper we extend this relationship to formulate a minimization problem based on the product of the determinants of all users' MSE matrices (PDetMSE), toward maximizing mutual information (and thus sum throughput) in the multiuser MIMO downlink. We show that maximizing the sum throughput under linear precoding is equivalent to the PDetMSE problem. This non-convex optimization problem is solved numerically, guaranteeing only local convergence. The performance of this approach is analyzed via comparison to the sum capacity and to existing approaches for linear precoding.

The remainder of this paper is organized as follows. Section II describes the system model used and states the assumptions made. Section III provides some background on the sum capacity of the MIMO broadcast channel and on the achievable sum rate under linear precoding, and Section III-C examines some existing schemes based on linear precoding. Section IV investigates the use of the product of MSE matrix

This research was supported in part by LG Electronics, Inc. of Korea, the National Science and Engineering Research Council of Canada (NSERC), and the Ontario Graduate Scholarships in Science and Technology (OGSST).

978-1-4244-2247-0/08/$25.00 ©2008 IEEE.

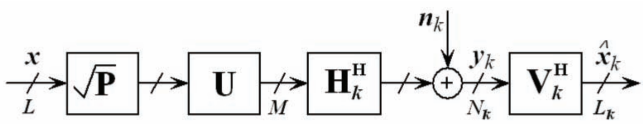

Fig. 1. Processing for user k in downlink.

determinants as the optimization criterion under a sum power constraint. Results of simulations testing the effectiveness of the proposed approach are presented in Section V. Finally, we draw conclusions in Section VI.

Notation: We use the following conventions: lower case italics, e.g., x, represent scalars while lower case boldface type is used for vectors (e.g., $\mathbf{x}$). Upper case italics, e.g., N, are used for constants and upper case boldface represents matrices, e.g., $\mathbf{X}$. Entries in vectors and matrices are denoted as $[\mathbf{x}]_i$ and $[\mathbf{X}]_{i,j}$. The superscripts ^{T} and ^{H} denote the transpose and Hermitian operators respectively. $\mathbb{E}[\cdot]$ represents the statistical expectation operator while $\mathbf{I}_N$ is the $N \times N$ identity matrix. $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_2$ denote the 1-norm (sum of entries) and Euclidean norm. $\mathrm{diag}(\mathbf{x})$ represents the diagonal matrix formed using the entries in vector $\mathbf{x}$, and $\mathrm{diag}[\mathbf{X}_1, \ldots, \mathbf{X}_k]$ is the block diagonal concatenation of matrices $\mathbf{X}_1, \ldots, \mathbf{X}_k$. $\mathbf{A} \succ \mathbf{0}$ and $\mathbf{B} \succeq \mathbf{0}$ indicate that $\mathbf{A}$ and $\mathbf{B}$ are positive definite and positive semidefinite matrices, respectively; $\mathbf{C} \succ \mathbf{D}$ states that the matrix $\mathbf{C} - \mathbf{D}$ is positive definite. Finally, $\mathcal{CN}(m, \sigma^2)$ denotes the complex Gaussian probability distribution with mean m and variance σ^2.

II. System Model Under Linear Precoding

The system under consideration, illustrated in Fig. 1, comprises a base station with M antennas transmitting to K decentralized users over flat wireless channels. User k is equipped with N_k antennas and receives L_k data streams from the base station. Thus, we have M transmit antennas transmitting a total of $L = \sum_{k=1}^{K} L_k$ symbols to K users, who together have a total of $N = \sum_{k=1}^{K} N_k$ receive antennas. The data symbols for user k are collected in the data vector $\mathbf{x}_k = [x_{k1}, x_{k2}, \ldots, x_{kL_k}]^T$ and the overall data vector is $\mathbf{x} = \left[\mathbf{x}_1^T, \mathbf{x}_2^T, \ldots, \mathbf{x}_K^T\right]^T$. User k's data streams are processed by the $M \times L_k$ transmit filter $\mathbf{U}_k = [\mathbf{u}_{k1}, \ldots, \mathbf{u}_{kL_k}]$ before being transmitted over the M antennas; $\mathbf{u}_{kj}$ is the precoder for stream j of user k, and has unit power $\|\mathbf{u}_{kj}\|_2 = 1$. Together, these individual precoders form the $M \times L$ global transmitter precoder matrix $\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_K]$. Let p_{kj} be the power allocated to stream j of user k and the downlink transmit power vector for user k be $\mathbf{p}_k = [p_{k1}, p_{k2}, \ldots, p_{kL_k}]^T$, with $\mathbf{p} = \left[\mathbf{p}_1^T, \ldots, \mathbf{p}_K^T\right]^T$. Define $\mathbf{P}_k = \mathrm{diag}\{\mathbf{p}_k\}$ and $\mathbf{P} = \mathrm{diag}\{\mathbf{p}\}$. The channel between the transmitter and user k is represented by the $N_k \times M$ matrix $\mathbf{H}_k^H$. The overall $N \times M$ channel matrix is $\mathbf{H}^H$, with $\mathbf{H} = [\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_K]$. The transmitter is assumed to know all the channels perfectly.

Based on this model, user k receives a length-N_k vector

$$\mathbf{y}_k = \mathbf{H}_k^H \mathbf{U} \sqrt{\mathbf{P}} \mathbf{x} + \mathbf{n}_k, \tag{1}$$

where $\mathbf{n}_k$ consists of the additive white Gaussian noise (AWGN) at the user's receive antennas with i.i.d. entries $[\mathbf{n}_k]_i \sim \mathcal{CN}(0, \sigma^2)$; that is, $\mathbf{R}_n = \mathbb{E}\left[\mathbf{n}_k \mathbf{n}_k^H\right] = \sigma^2 \mathbf{I}_{N_k}$. This assumption of independent AWGN is made without loss of generality; for coloured and/or correlated noise covariance matrices $\mathbf{R}_n$, a linear transformation of $\mathbf{R}_n$ to the identity matrix can be incorporated into the channel matrices $\mathbf{H}_j$. To estimate its L_k symbols $\mathbf{x}_k$, user k processes $\mathbf{y}_k$ with its $L_k \times N_k$ decoder matrix $\mathbf{V}_k^H$ resulting in

$$\hat{\mathbf{x}}_k^{DL} = \mathbf{V}_k^H \mathbf{H}_k^H \mathbf{U} \sqrt{\mathbf{P}} \mathbf{x} + \mathbf{V}_k^H \mathbf{n}_k, \tag{2}$$

where the superscript ^{DL} indicates the downlink. The global receive filter $\mathbf{V}^H$ is a block diagonal matrix of dimension $L \times N$, $\mathbf{V} = \mathrm{diag}[\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_K]$, where each $\mathbf{V}_k = [\mathbf{v}_{k1}, \ldots, \mathbf{v}_{kL_k}]$.

We assume that the modulated data symbols $\mathbf{x}$ are drawn from a PSK constellation where each data symbol x_i has power $|x_i|^2 = 1$. Furthermore, the data symbols are independent of each other, so that $\mathbb{E}\left[\mathbf{x}\mathbf{x}^H\right] = \mathbf{I}_L$, and are also independent of the noise.

III. Linear Precoding and Sum Broadcast Capacity

A. Broadcast Channel Sum Capacity

Information theoretic approaches characterize the sum capacity of the multiuser MIMO downlink by solving the sum capacity of the equivalent uplink multiple access channel (MAC) and applying a duality result [1], [2]. The MAC capacity is found by solving the problem in (3), where transmit covariance matrices $\mathbf{\Sigma}_k$ are designed for each mobile user k, subject to a sum power constraint of $P_{\max}$:

$$\begin{aligned} R_{\text{sum}} \quad = \quad & \max_{\mathbf{\Sigma}_k} \log\det\left(\mathbf{I} + \frac{1}{\sigma^2}\sum_{k=1}^{K} \mathbf{H}_k \mathbf{\Sigma}_k \mathbf{H}_k^H\right) \\ \text{s.t.} \quad & \mathbf{\Sigma}_k \succeq \mathbf{0}, \quad k = 1, \ldots, K \\ & \sum_{k=1}^{K} \mathrm{tr}\left[\mathbf{\Sigma}_k\right] \leq P_{\max}. \end{aligned} \tag{3}$$

One key feature of this optimization problem is that it is convex in $\mathbf{\Sigma}_k$, and can be solved efficiently using well established techniques.

B. Sum Rate under Linear Precoding

When linear precoding is employed, the convex solution above does not generally exist. If each user transmits with covariance matrix $\mathbf{\Sigma}_k$, the achievable rate for user k under linear precoding is

$$R_k \quad = \quad \log \frac{\det\left(\sum_{j=1}^{K} \mathbf{H}_k^H \mathbf{\Sigma}_j \mathbf{H}_k + \sigma^2 \mathbf{I}\right)}{\det\left(\sum_{j \neq k} \mathbf{H}_k^H \mathbf{\Sigma}_j \mathbf{H}_k + \sigma^2 \mathbf{I}\right)}. \tag{4}$$

Under the system model described in Section II, user k transmits with covariance matrix $\mathbf{\Sigma}_k = \mathbf{U}_k \mathbf{P}_k \mathbf{U}_k^H$.

The achievable rate under linear precoding is therefore

$$\begin{aligned} R_k^{\mathbf{LP}} &= \log \frac{\det\left(\sum_{j=1}^{K} \mathbf{H}_k^H \mathbf{U}_j \mathbf{P}_j \mathbf{U}_j^H \mathbf{H}_k + \sigma^2 \mathbf{I}\right)}{\det\left(\sum_{j\neq k} \mathbf{H}_k^H \mathbf{U}_j \mathbf{P}_j \mathbf{U}_j^H \mathbf{H}_k + \sigma^2 \mathbf{I}\right)} \\ &= \log \frac{\det \mathbf{J}_k}{\det \mathbf{R}_{N+I,k}} \end{aligned}$$

where $\mathbf{J}_k = \mathbf{H}_k^H \mathbf{U}\mathbf{P}\mathbf{U}^H \mathbf{H}_k + \sigma^2 \mathbf{I}$ and $\mathbf{R}_{N+I,k} = \mathbf{J}_k - \mathbf{H}_k^H \mathbf{U}_k \mathbf{P}_k \mathbf{U}_k^H \mathbf{H}_k$ are the received signal covariance matrix and the noise-plus-interference covariance matrix at user k, respectively.

The rate maximization problem with sum power constraint under linear precoding can then be formulated as

$$\begin{aligned} (\mathbf{U}, \mathbf{P}) &= \arg\max_{\mathbf{U},\mathbf{P}} \sum_{k=1}^{K} \log \frac{\det \mathbf{J}_k}{\det \mathbf{R}_{N+I,k}} \\ \text{s.t.} \quad & \|\mathbf{u}_{kj}\|_2 = 1, \quad k = 1, \ldots, K, \quad j = 1, \ldots, L_k \\ & p_{kj} \geq 0, \quad k = 1, \ldots, K, \quad j = 1, \ldots, L_k \\ & \|\mathbf{p}\|_1 = \sum_{k=1}^{K} \sum_{j=1}^{L_k} p_{kj} \leq P_{\max}. \end{aligned} \tag{5}$$

A scalarized version of the same problem considers the user's own data streams $l \neq j$ as self-interference (in addition to the multiuser interference). Under the assumption that the noise-plus-interference is approximately Gaussian (valid for a sufficient number of interferers by the central limit theorem), we can write the scalar rate for user k's substream j as

$$R_{k,j}^{\mathbf{LP}} = \log\left(1 + \gamma_{kj}^{\mathbf{DL}}\right),$$

where

$$\gamma_{kj}^{\mathbf{DL}} = \frac{\mathbf{v}_{kj}^H \mathbf{H}_k^H \mathbf{u}_{kj} p_{kj} \mathbf{u}_{kj}^H \mathbf{H}_k \mathbf{v}_{kj}}{\mathbf{v}_{kj}^H \mathbf{J}_{kj} \mathbf{v}_{kj}}, \tag{6}$$

and $\mathbf{J}_{kj} = \mathbf{J}_k - \mathbf{H}_k^H \mathbf{u}_{kj} p_{kj} \mathbf{u}_k j^H \mathbf{H}_k$ is the noise-plus-interference covariance matrix as seen by stream j belonging to user k.

The scalar rate maximization problem with sum power constraint under linear precoding can thus be written as

$$\begin{aligned} (\mathbf{U}, \mathbf{P}) &= \arg\max_{\mathbf{U},\mathbf{P}} \sum_{k=1}^{K} \sum_{j=1}^{L_k} \log\left(1 + \gamma_{kj}\right) \\ \text{s.t.} \quad & \|\mathbf{u}_{kj}\|_2 = 1, \quad k = 1, \ldots, K, \quad j = 1, \ldots, L_k \\ & p_{kj} \geq 0, \quad k = 1, \ldots, K, \quad j = 1, \ldots, L_k, \\ & \|\mathbf{p}\|_1 = \sum_{k=1}^{K} \sum_{j=1}^{L_k} p_{kj} \leq P_{\max}. \end{aligned} \tag{7}$$

C. Existing Schemes for Linear Precoding

1) Orthogonalization-Based Methods: In block diagonalization, each user's precoder is selected so that it is orthogonal to the channels of all other users. The precoder matrices are required to satisfy the following orthogonality constraints:

$$\mathbf{H}_k^H \mathbf{U}_j = \mathbf{0}, \quad k = 1, \ldots, K; \quad j \neq k. \tag{8}$$

By applying this orthogonalization, the multiuser downlink is transformed into a set of K parallel single-user MIMO effective channels, $\mathbf{G}_k = \mathbf{H}_k^H \mathbf{U}_k$. Satisfying all of the equations in (8) requires at least $M \geq N - \min(N_1, \ldots, N_k)$ transmit antennas; conversely, satisfying the orthogonality constraints consumes $N - N_k$ degrees of freedom for user k. The resulting effective channels $\mathbf{G}_k$ are statistically equivalent to i.i.d. complex Gaussian $(M - N + N_k) \times N_k$ channels [11]. The maximum sum rate that can be achieved under the BD precoder by convex waterfilling over the parallel Gaussian channels is then

$$\begin{aligned} R_{\max}^{BD} &= \max_{\mathbf{P}_k} \sum_{k=1}^{K} \log\det\left(\mathbf{I} + \frac{1}{\sigma^2} \mathbf{G}_k \mathbf{P}_k \mathbf{G}_k^H\right) \\ & \text{s.t.} \quad \sum_{k=1}^{K} \mathbf{P}_k \leq P_{\max}. \end{aligned} \tag{9}$$

An analogous formulation exists for zero-forcing, where the precoder for each substream is orthogonal to all subchannels other than its own. As in (6), each data stream treats all other data streams as interference. The orthogonality constraints are:

$$\begin{aligned} \mathbf{h}_{kn}^H \mathbf{u}_{kl} &= 0, \quad k = 1, \ldots, K; \quad l \neq n \\ \mathbf{h}_{kn}^H \mathbf{u}_{jl} &= 0, \quad k = 1, \ldots, K; \quad j \neq k. \end{aligned} \tag{10}$$

The resulting N effective channels are scalar, $g_{kj} = \mathbf{h}_{kj}^H \mathbf{u}_{kj}$, and are each statistically equivalent to an $(M - N + 1) \times 1$ i.i.d. complex Gaussian vector channel in a single-user scenario. The maximum sum rate under zero-forcing can again be found by convex waterfilling over parallel channels:

$$\begin{aligned} R_{\max}^{ZF} &= \max_{p_{kj}} \sum_{k=1}^{K} \sum_{j=1}^{N_k} \log\left(1 + \frac{1}{\sigma^2} p_{kj} \left|g_{kj}\right|^2\right) \\ & \text{s.t.} \quad \sum_{k=1}^{K} \sum_{j=1}^{N_k} p_{kj} \leq P_{\max}. \end{aligned} \tag{11}$$

2) Direct Optimization: Both block diagonalization and zero-forcing consume available degrees of freedom to satisfy orthogonality constraints, thus reducing the diversity order of the effective channel and limiting the number of receive antennas that can be present for a fixed number of transmit antennas. This restriction has motivated research on direct optimization of the sum-rate under linear precoding [12], and on maximization of the SINR expression (6) using uplink-downlink duality [13], [14]. In each of these papers, approximations and iterative solutions are used to solve (7), while avoiding the antenna constraints associated with orthogonalization. Since optimization of (7) is a non-convex problem, these solutions only converge to local minima (if at all), and do not admit closed-form solutions.

Recently, [15], [19] have independently derived a relationship between the sum rate under linear precoding (7) and the product of mean squared errors (PMSE). By formulating the optimization problem as a function of the MSEs $\epsilon_{kj} = \mathbb{E}\left[\left|\hat{x}_{kj} - x_{kj}\right|^2\right]$, an iterative solution can be designed using the alternating optimization framework in a manner similar to minimization the sum of mean squared errors (SMSE) [17], [18]. The PMSE minimization problem is written as

$$\begin{aligned} (\mathbf{U},\mathbf{P}) &= \arg\min_{\mathbf{U},\mathbf{P}} \prod_{k=1}^{K}\prod_{j=1}^{L_k} \epsilon_{kj} \\ \text{s.t.} &\quad \|\mathbf{u}_{kj}\|_2 = 1, \quad k=1,\ldots,K, \quad j=1,\ldots,L_k \\ &\quad u_{kj} \geq 0, \qquad k=1,\ldots,K, \quad j=1,\ldots,L_k \\ &\quad \|\mathbf{p}\|_1 = \sum_{k=1}^{K}\sum_{j=1}^{L_k} p_{kj} \leq P_{\max}, \end{aligned} \tag{12}$$

and is directly equivalent to the sum rate maximization problem (7).

IV. Product of MSE Matrix Determinants

This section presents the core contribution of this paper. In [21], the single-user rate maximization problem using linear precoding is solved by minimizing the determinant of the MSE matrix. The minimum determinant solution is equivalent to the PMSE solution for the single user case, as the MSE matrix can be diagonalized using a unitary transformation of the precoder matrix. Since this diagonalization is not possible in the multiuser case, the minimization of the PMSE and of the product of MSE matrix determinants (PDetMSE) yield different solutions. Furthermore, treating each user's own data streams as interference (as in PMSE) is sub-optimal compared to joint optimization over all of the user's data streams. In this section, we investigate this problem and develop the relationship between the MSE matrix determinants and the maximum achievable rate under linear precoding.

First, consider the downlink MSE matrix under linear MMSE decoding with receive matrices $\mathbf{V}_k$,

$$\begin{aligned} \mathbf{V}_k &= \left(\mathbf{H}_k^H\mathbf{U}\mathbf{P}\mathbf{U}^H\mathbf{H}_k + \sigma^2\mathbf{I}\right)^{-1}\mathbf{H}_k^H\mathbf{U}_k\sqrt{\mathbf{P}_k} \\ &= \mathbf{J}_k^{-1}\mathbf{H}_k^H\mathbf{U}_k\sqrt{\mathbf{P}_k}. \end{aligned} \tag{13}$$

When using this receive matrix, the MSE matrix achieved by user k in the downlink is

$$\begin{aligned} \mathbf{E}_k^{DL} &= \mathbb{E}\left[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^H\right] && (14)\\ &= \mathbf{I}_{L_k} - \sqrt{\mathbf{P}_k}\mathbf{U}_k^H\mathbf{H}_k\mathbf{J}_k^{-1}\mathbf{H}_k^H\mathbf{U}_k\sqrt{\mathbf{P}_k} && (15) \end{aligned}$$

Consider the following optimization problem which minimizes the product of the determinants of the downlink MSE matrices under a sum power constraint:

$$\begin{aligned} (\mathbf{U},\mathbf{P}) &= \arg\min_{\mathbf{U},\mathbf{P}} \prod_{k=1}^{K} \det\mathbf{E}_k^{DL} \\ \text{s.t.} &\quad \|\mathbf{u}_{kj}\|_2 = 1, \quad k=1,\ldots,K, \quad j=1,\ldots,L_k \\ &\quad p_{kj} \geq 0, \qquad k=1,\ldots,K, \quad j=1,\ldots,L_k \\ &\quad \|\mathbf{p}\|_1 = \sum_{k=1}^{K}\sum_{j=1}^{L_k} p_{kj} \leq P_{\max}. \end{aligned} \tag{16}$$

Theorem 1: Under linear MMSE decoding at the base station, the sum rate maximization problem in (5) and the PDetMSE minimization problem in (16) are equivalent.

Proof: The determinant of the downlink MSE matrix can be written as

$$\begin{aligned} \det\mathbf{E}_k^{DL} &= \det\left(\mathbf{I}_{L_k} - \mathbf{H}_k^H\mathbf{U}_k\mathbf{P}_k\mathbf{U}_k^H\mathbf{H}_k\mathbf{J}_k^{-1}\right) && (17)\\ &= \det\left[\left(\mathbf{J}_k - \mathbf{H}_k^H\mathbf{U}_k\mathbf{P}_k\mathbf{U}_k^H\mathbf{H}_k\right)\mathbf{J}_k^{-1}\right] && (18)\\ &= \det\left[\mathbf{R}_{N+I,k}\mathbf{J}_k^{-1}\right] && (19)\\ &= \frac{\det\mathbf{R}_{N+I,k}}{\det\mathbf{J}_k}, && (20) \end{aligned}$$

where (17) follows from (15) since $\det(\mathbf{I}+\mathbf{AB}) = \det(\mathbf{I}+\mathbf{BA})$ when $\mathbf{A}$ and $\mathbf{B}$ have appropriate dimensions. We then see the relationship to (5),

$$\begin{aligned} \log\det\mathbf{E}_k^{DL} &= -\log\frac{\det\mathbf{J}_k}{\det\mathbf{R}_{N+I,k}} && (21)\\ &= -R_k^{\mathbf{LP}}. && (22) \end{aligned}$$

With this result, we can see that under MMSE reception using $\mathbf{V}_k$ as defined in (13), minimizing the determinant of the MSE matrix $\mathbf{E}_k^{DL}$ is equivalent to maximizing the achievable rate for user k. It follows that minimizing the product of MSE matrix determinants over all users is equivalent to sum-rate maximization,

$$\begin{aligned} \min\prod_{k=1}^{K}\det\mathbf{E}_k^{DL} &\equiv \min\sum_{k=1}^{K}\log\det\mathbf{E}_k^{DL} && (23)\\ &\equiv \max\sum_{k=1}^{K}R_k^{\mathbf{LP}}. && (24) \end{aligned}$$

where (23) holds since since $\log(\cdot)$ is a monotonically increasing function of its argument. ■

The covariance matrices $\mathbf{J}_k$ and $\mathbf{R}_{N+I,k}$ in the MSE matrix $\mathbf{E}_k$ are each functions of all precoder and power allocation matrices. Thus, the sum rates R_k for each user k (and the sum rate for all users) are coupled across users. As such, finding $\mathbf{U}$ and $\mathbf{P}$ jointly or finding only the power allocation $\mathbf{P}$ for a fixed $\mathbf{U}$ are both non-convex problems and are just as difficult to solve as the rate maximization problem.

In the sum capacity and SMSE/PMSE problems, the problem of non-convexity is addressed by solving an equivalent virtual uplink formulation and applying a duality-based transformation. Unfortunately, the sum rate expression under linear precoding in the virtual uplink is nearly identical to that derived above for the downlink, and does not admit a cancellation or grouping of terms to decouple the problem across users. Moreover, there does not appear to be a duality between the product of MSE determinants in the uplink and downlink. Thus, while we do believe that the uplink power allocation subproblem may be a geometric programming (GP) problem (as shown for the PMSE case in [15]), an iterative solution based on uplink-downlink duality can not be applied.

Direct solution of the non-convex downlink problem for minimizing the product of MSE matrix determinants requires finding a complex $M \times L$ precoder matrix. We can apply sequential quadratic programming (SQP) [22] to solve the PDetMSE minimization problem. SQP solves successive approximations of a constrained optimization problem and is guaranteed to converge to the optimum value for convex problems; however, in the case of this non-convex optimization problem, SQP can only guarantee convergence to a local minimum. This computationally intensive method is clearly not a desirable method for finding a practical precoder, especially when one of our major motivations for using linear precoding is reducing transmitter complexity; however, in this case, SQP seems to be the only available option. We do not suggest that this method be practically implemented; rather, we use it to illustrate the optimality of PDetMSE formulation as the theoretical upper bound on performance, in contrast to the previously proposed PMSE.

V. Numerical Examples

In this section, we present simulation results to illustrate the performance of the proposed algorithm. In all cases, the fading channel is modelled as flat and Rayleigh, with i.i.d. channel coefficients distributed as $\mathcal{CN}(0,1)$. The examples use a maximum transmit power of $P_{\max} = 1$; SNR is controlled by varying the receiver noise power σ^2. The transmitter is assumed to have perfect knowledge of the channel matrix $\mathbf{H}$.

We compare the sum rate achievable using linear precoding and the information theoretic capacity of the BC. That is, we consider the spectral efficiency (measured in bps/Hz) that could be achieved under ideal transmission by drawing transmit symbols from a Gaussian codebook. Figure 2 illustrates how the proposed schemes perform when compared to the sum capacity for the broadcast channel (i.e. using dirty paper coding (DPC) [6]) and to traditional linear precoding methods based on channel orthogonalization, i.e., block diagonalization (BD) and zero forcing (ZF) [11]. Simulation results for the DPC, BD, and ZF plots were obtained by using the `cvx` optimization package [23], [24]. This simulation models a $K = 2$ user system with $M = 4$ transmit antennas and $N_k = 2$ or $N_k = 4$ receive antennas per user. The plot is generated using 30000 channel realizations, with 5000 data symbols per channel realization. This example also serves to illustrate the flexibility of the PMSE and PDetMSE schemes since BD and ZF cannot be used with $N_k = 4$ receive antennas. Note that curves for THP can not be included for comparison, as the modulo and shaping losses from the DPC sum capacity are fundamentally related to THP's nonlinear modulation scheme.

In Fig. 2, we see a negligible difference in performance between the PMSE and PDetMSE algorithms. Detailed views in Fig. 3 and Fig. 4 illustrate that the PDetMSE solution does marginally outperform the PMSE-based solution; however, these plots suggests that even though joint processing may allow for increased throughput, the gains are small and are probably not worth the greatly increased computational complexity. The PMSE and PDetMSE algorithms also both demonstrate a divergence in performance when compared to the theoretical DPC bound at higher SNR. This drop in spectral efficiency may be caused by the non-convexity of the optimization problem causing convergence to local minima. Nonetheless, these schemes still maintain a higher spectral efficiency than the orthogonalization based schemes for $N_k = 2$. Furthermore, the gap between the DPC bound and the PDetMSE/PMSE precoders is only 0.6 dB for $N_k = 4$, where BD and ZF schemes *can not be applied* due to constraints on the number of antennas.

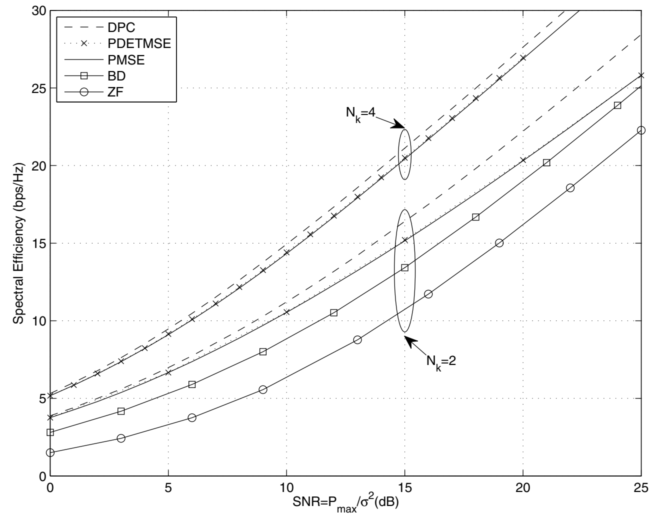

Fig. 2. Comparing PDetMSE, PMSE, DPC and orthogonalization–based methods

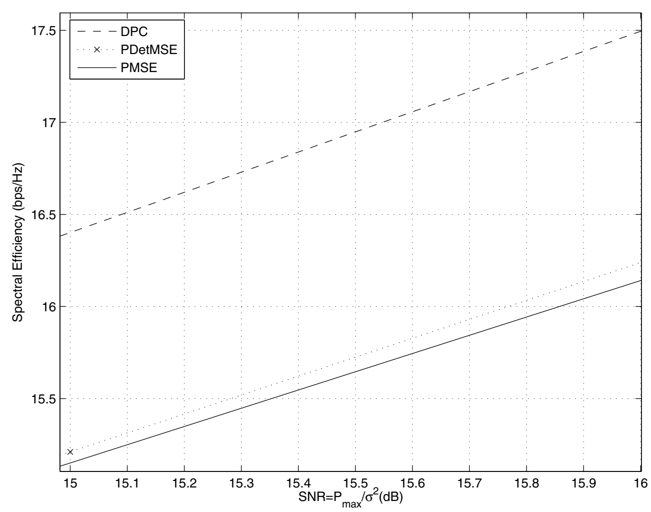

Fig. 3. Detailed view of DPC vs PDetMSE/PMSE, $N_k = 2$

VI. Conclusions

In this paper, we have considered the problem of designing the theoretically optimal linear precoder that maximizes sum throughput in the multiuser MIMO downlink. We have compared the sum rate performance of linear precoding schemes to the sum capacity in the general MIMO downlink, without

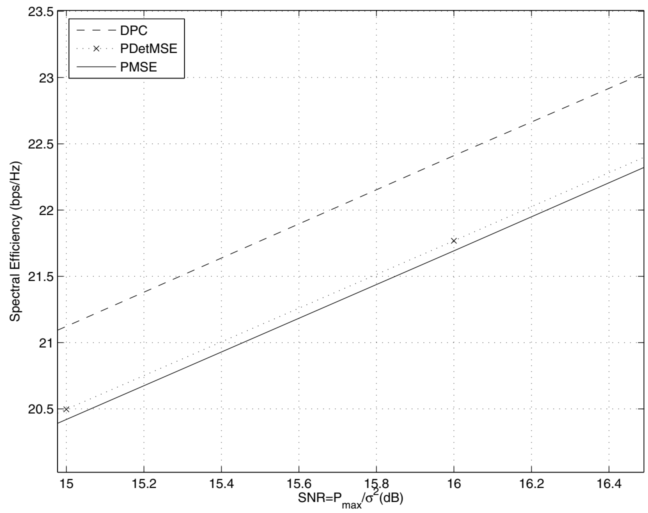

Fig. 4. Detailed view of DPC vs PDetMSE/PMSE, $N_k = 4$

imposing constraints on the number of users, base station antennas, or mobile antennas. Previous work in linear precoding for the multiuser downlink has focused on orthogonalizing approaches (which suffer from reduced diversity order and restrictive antenna constraints), and on the scalarized form of the sum rate maximization problem. We have shown that the optimal problem formulation for maximizing the sum rate under linear precoding uses joint MSE processing via minimization of the product of MSE matrix determinants. Simulations demonstrate that the proposed scheme marginally outperforms the PMSE based solution, but at the expense of greatly increased computational complexity.

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