

Minimizing Sum-MSE Implies Identical Downlink and Dual Uplink Power Allocations

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Abstract—Minimizing the sum of mean squared errors using linear transceivers under a sum power constraint in the multiuser downlink is a non-convex problem. Existing algorithms exploit an uplink-downlink duality and transform the solution of a convex problem in the virtual uplink back to the downlink. In this letter, we analyze the optimality criteria for the power allocation subproblem in the virtual uplink, and demonstrate that the optimal solution leads to identical power allocations in the downlink and virtual uplink, thus extending the known duality results and permitting a reduction in the computational complexity of existing algorithms.

Index Terms—MIMO systems, optimization methods, least mean square methods

I. INTRODUCTION

MINIMIZATION of the sum of mean squared errors (sum-MSE) under a sum power constraint using linear precoding and decoding in the multiple-input, multiple-output (MIMO) multiuser downlink is a well-studied problem. The downlink sum-MSE minimization problem is non-convex; thus, it is not possible to directly solve the problem in its original downlink formulation. In [1]–[4], an equivalent problem is solved using the *virtual uplink* model, wherein the roles of transmitter and receiver are exchanged. In the virtual uplink, the receiver is the Wiener filter and the power allocation is convex.

The equivalence of the downlink and virtual uplink problems are enabled by an *uplink-downlink duality* result for the MSE of each data stream. Duality results for linear precoding systems were first presented for signal-to-interference-plus-noise ratios (SINR) in [5] with single-antenna receivers. This work was later extended to MSEs and systems with multiple receive antennas in [1], [2] and subsequently generalized in [6]. A necessary and common feature to all of this work is a transformation of the power allocated to each data stream in the virtual uplink to the downlink, while achieving the same SINR/MSE in each stream in both systems. This transformation requires the solution of a matrix equation.

In this letter, we use the Karush-Kuhn-Tucker (KKT) conditions for the power allocation subproblem in the virtual uplink to show that at the optimal point, the powers allocated to each data stream in both the downlink and virtual uplink are identical. This result extends the known dualities in the multiuser MIMO case. Furthermore, this also eliminates the need for the uplink-to-downlink power transformation.

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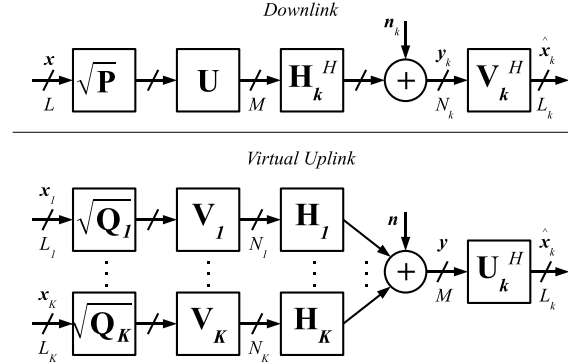


Fig. 1. Processing for user k in downlink and virtual uplink.

Section II describes the system model and existing algorithms for minimizing the sum-MSE using uplink-downlink duality. In Section III, we present the KKT conditions for the virtual uplink power allocation subproblem, and use the resulting expressions to prove the equality of the downlink and virtual uplink power allocations. We present our conclusions in Section IV.

II. BACKGROUND

A. System Model with Linear Precoding

In the linear precoding system, illustrated in Fig. 1, a base station with M antennas transmits to K decentralized mobile users with N_k antennas each over flat wireless channels. The channel between the transmitter and user k is represented by the $N_k \times M$ matrix \mathbf{H}_k^H , and the overall $N \times M$ channel matrix is \mathbf{H}^H , with $\mathbf{H} = [\mathbf{H}_1, \dots, \mathbf{H}_K]$. User k receives L_k data symbols $\mathbf{x}_k = [x_{k1}, \dots, x_{kL_k}]^T$ from the base station, and the vector $\mathbf{x} = [\mathbf{x}_1^T, \dots, \mathbf{x}_K^T]^T$ comprises independent symbols with unit average energy ($\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{I}_L$, where $L = \sum_{k=1}^K L_k$). User k 's data streams are precoded by the $M \times L_k$ transmit filter $\mathbf{U}_k = [\mathbf{u}_{k1}, \dots, \mathbf{u}_{kL_k}]$, where \mathbf{u}_{kj} is the precoding beamformer for stream j of user k with $\|\mathbf{u}_{kj}\| = 1$. These individual precoders are combined in the $M \times L$ global transmitter precoder matrix $\mathbf{U} = [\mathbf{U}_1, \dots, \mathbf{U}_K]$. Power is allocated to user k 's data streams in the vector $\mathbf{p}_k = [p_{k1}, \dots, p_{kL_k}]^T$ and $\mathbf{P}_k = \text{diag}[\mathbf{p}_k]$; we define the downlink power allocation matrix as $\mathbf{P} = \text{diag}\{\mathbf{p}_1^T, \dots, \mathbf{p}_K^T\}$. Based on this model, user k receives a length- N_k vector $\mathbf{y}_k^{DL} = \mathbf{H}_k^H \mathbf{U} \sqrt{\mathbf{P}} \mathbf{x} + \mathbf{n}_k$, where the superscript DL indicates the downlink, and $\mathbf{n}_k \sim \mathcal{CN}(0, \sigma^2 \mathbf{I}_{N_k})$ consists of zero-mean white Gaussian noise. To estimate its L_k symbols \mathbf{x}_k , user k

applies the $L_k \times N_k$ receive filter \mathbf{V}_k^H , yielding the estimated symbols $\hat{\mathbf{x}}_k^{DL} = \mathbf{V}_k^H \mathbf{H}_k^H \mathbf{U} \sqrt{\mathbf{P}} \mathbf{x} + \mathbf{V}_k^H \mathbf{n}_k$.

In order to minimize the sum-MSE in the multiuser MIMO downlink, we use the virtual uplink, also illustrated in Fig. 1, where each matrix is replaced by its conjugate transpose. In this transformed system, we imagine transmissions from mobile user k that propagate via the *flipped channel* \mathbf{H}_k to the base station. The transmit and receive filters for user k become \mathbf{V}_k and \mathbf{U}_k^H respectively, with normalized precoding beamformers; i.e., $\|\mathbf{v}_{kj}\| = 1$. Power is allocated to user k 's data streams as $\mathbf{q}_k = [q_{k1}, \dots, q_{kL_k}]^T$, with $\mathbf{Q}_k = \text{diag}\{\mathbf{q}_k\}$ and $\mathbf{Q} = \text{diag}\{\mathbf{q}_1^T, \dots, \mathbf{q}_K^T\}$. The received symbol vector at the base station and the estimated symbol vector for user k are $\mathbf{y}^{UL} = \sum_{i=1}^K \mathbf{H}_i \mathbf{V}_i \sqrt{\mathbf{Q}_i} \mathbf{x}_i + \mathbf{n}$ and $\mathbf{x}_k^{UL} = \sum_{i=1}^K \mathbf{U}_k^H \mathbf{H}_i \mathbf{V}_i \sqrt{\mathbf{Q}_i} \mathbf{x}_i + \mathbf{U}_k^H \mathbf{n}$, respectively, with zero-mean white Gaussian noise $\mathbf{n} \sim \mathcal{CN}(0, \sigma^2 \mathbf{I}_M)$.

B. Minimum Sum-MSE Multiuser MIMO Linear Precoding

1) *Convex Minimum Sum-MSE Precoder Design*: The MSE matrix for user k in the downlink using arbitrary precoder and decoder matrices can be written as

$$\begin{aligned} \mathbf{E}_k^{DL} &= \mathbb{E} \left[(\hat{\mathbf{x}}_k^{DL} - \mathbf{x}_k) (\hat{\mathbf{x}}_k^{DL} - \mathbf{x}_k)^H \right] \\ &= \mathbf{V}_k^H \mathbf{J}_k \mathbf{V}_k - \mathbf{V}_k^H \mathbf{H}_k^H \bar{\mathbf{U}}_k - \bar{\mathbf{U}}_k^H \mathbf{H}_k \mathbf{V}_k + \mathbf{I}_{L_k}, \end{aligned} \quad (1)$$

where $\mathbf{J}_k = \mathbf{H}_k^H \mathbf{U} \mathbf{P} \mathbf{U}^H \mathbf{H}_k + \sigma^2 \mathbf{I}$, $\bar{\mathbf{U}}_k = \mathbf{U}_k \sqrt{\mathbf{P}_k}$, and data and noise terms are assumed to be independent. The individual MSE terms are minimized using the minimum MSE (MMSE) receiver $\tilde{\mathbf{V}}_k^H = \bar{\mathbf{U}}_k^H \mathbf{H}_k \mathbf{J}_k^{-1}$. The resulting MMSE matrix is $\bar{\mathbf{E}}_k^{DL} = \mathbf{I}_{L_k} - \bar{\mathbf{U}}_k^H \mathbf{H}_k \mathbf{J}_k^{-1} \mathbf{H}_k^H \bar{\mathbf{U}}_k$, and the minimum sum-MSE for any choice of $\bar{\mathbf{U}}_k$ is $\text{SMSE}^{DL} = \sum_{k=1}^K \text{tr} \left[\bar{\mathbf{E}}_k^{DL} \right]$.

The problem of finding the sum-MSE minimizing precoders and power allocations in the downlink under a sum power constraint $\text{tr}[\mathbf{P}] \leq P_{\max}$ is non-convex due to the cross-coupling introduced by the presence of all $\bar{\mathbf{U}}_i$ terms in every \mathbf{J}_k . However, a transformation to the virtual uplink gives rise to several equivalent problems that can be solved using convex optimization. The set of virtual uplink minimum sum-MSE precoders and power allocations $\{(\mathbf{V}_k, \mathbf{q}_k), k = 1, \dots, K\}$ can be found jointly, by finding the optimum covariance matrices $\mathbf{R}_k = \mathbf{V}_k^H \mathbf{Q}_k \mathbf{V}_k$ and applying Cholesky or eigen-decomposition [4]. An alternative approach finds the optimum precoders \mathbf{V}_k and power allocations \mathbf{q}_k in an iterative manner [2], [4]. The convexity of these problems originates from the decoupling of users in the virtual uplink. The MMSE matrix $\bar{\mathbf{E}}_k^{UL}$ for user k is found using the MMSE receiver $\tilde{\mathbf{U}}_k^H = \tilde{\mathbf{V}}_k^H \mathbf{H}_k^H \mathbf{J}^{-1}$,

$$\bar{\mathbf{E}}_k^{UL} = \mathbf{I}_{L_k} - \tilde{\mathbf{V}}_k^H \mathbf{H}_k^H \mathbf{J}^{-1} \mathbf{H}_k \tilde{\mathbf{V}}_k, \quad (2)$$

with $\tilde{\mathbf{V}}_k = \mathbf{V}_k \sqrt{\mathbf{Q}_k}$ and $\mathbf{J} = \sum_{k=1}^K \mathbf{H}_k \tilde{\mathbf{V}}_k \tilde{\mathbf{V}}_k^H \mathbf{H}_k^H + \sigma^2 \mathbf{I}_M$. The resulting minimum sum-MSE is

$$\begin{aligned} \text{SMSE}^{UL} &= \sum_{k=1}^K L_k - \text{tr} \left[\mathbf{J}^{-1} \sum_{k=1}^K \mathbf{H}_k \tilde{\mathbf{V}}_k \tilde{\mathbf{V}}_k^H \mathbf{H}_k^H \right] \\ &= L - M + \sigma^2 \text{tr} \left[\mathbf{J}^{-1} \right], \end{aligned} \quad (3)$$

which follows from $\text{tr}[\mathbf{AB}] = \text{tr}[\mathbf{BA}]$, linearity of the trace operator, and the definition of \mathbf{J} . Minimizing the sum-MSE

thus only requires minimization of $\text{tr}[\mathbf{J}^{-1}]$, which is convex for both the power allocation subproblem in \mathbf{q}_k and the joint precoder design problem in covariance matrices \mathbf{R}_k . In this letter, we consider the former optimization problem, which is formally stated as

$$\begin{aligned} (\tilde{q}_1, \dots, \tilde{q}_L) &= \arg \min_{q_1, \dots, q_L} \text{tr} \left[\left(\sum_{l=1}^L q_l \tilde{\mathbf{h}}_l \tilde{\mathbf{h}}_l^H + \sigma^2 \mathbf{I}_M \right)^{-1} \right] \\ \text{s.t. } q_l &\geq 0 \quad l = 1, \dots, L, \quad \sum_{l=1}^L q_l \leq P_{\max}, \end{aligned} \quad (4)$$

where we have defined the effective channel $\tilde{\mathbf{H}} = [\mathbf{H}_1 \mathbf{V}_1, \dots, \mathbf{H}_K \mathbf{V}_K] = [\tilde{\mathbf{h}}_1, \dots, \tilde{\mathbf{h}}_L]$. Note that the columns in $\tilde{\mathbf{H}}$ refer to the effective channel vectors for each individual data stream $l = 1, \dots, L$.

2) *Uplink-Downlink Duality*: The duality result in [2], [4] shows that the achievable set of SINRs and MSEs for all individual data streams under a sum power constraint are equal in both the downlink and virtual uplink. If a selected set of target SINRs γ_l is achievable under the specified power constraint, the corresponding downlink and virtual uplink power allocations that satisfy the SINR targets are

$$\begin{aligned} \mathbf{p} &= \sigma^2 (\mathbf{D}^{-1} - \Psi)^{-1} \mathbf{1}_L \\ \mathbf{q} &= \sigma^2 (\mathbf{D}^{-1} - \Psi^T)^{-1} \mathbf{1}_L, \end{aligned} \quad (5)$$

where $\mathbf{1}_L$ is the length- L vector consisting of all ones,

$$\mathbf{D} = \text{diag} \left\{ \frac{\gamma_1}{|\tilde{\mathbf{h}}_1^H \mathbf{u}_1|^2}, \dots, \frac{\gamma_L}{|\tilde{\mathbf{h}}_L^H \mathbf{u}_L|^2} \right\}, \quad (6)$$

and

$$[\Psi]_{ij} = \begin{cases} |\tilde{\mathbf{h}}_i^H \mathbf{u}_j|^2 & i \neq j \\ 0 & i = j \end{cases}. \quad (7)$$

We have relabelled the columns of $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_L]$ as we did with $\tilde{\mathbf{H}}$ above to refer to the beamforming vectors for each data stream.

III. EQUALITY OF DOWNLINK AND UPLINK POWER ALLOCATIONS

Based on (5), we see that $\Psi = \Psi^T$ is a sufficient condition for the equality of \mathbf{p} and \mathbf{q} . We now proceed to prove that this transpose symmetry indeed applies for arbitrary virtual uplink precoders \mathbf{V}_k (with unit norm columns) as long as the optimum power allocation $\tilde{\mathbf{q}} = [\tilde{q}_1, \dots, \tilde{q}_L]$ satisfying (4) and the corresponding MMSE receive beamformers \mathbf{u}_l are used.

A. KKT Conditions for MMSE Precoding

From the objective and constraint functions in (4), we define the Lagrangian

$$\begin{aligned} \mathcal{L}(\mathbf{q}, \mu) &= \text{tr} \left[\left(\sum_{l=1}^L q_l \tilde{\mathbf{h}}_l \tilde{\mathbf{h}}_l^H + \sigma^2 \mathbf{I}_M \right)^{-1} \right] \\ &+ \mu_{\text{sum}} \left(\sum_{l=1}^L q_l - P_{\max} \right) - \sum_{l=1}^L \mu_l q_l \end{aligned} \quad (8)$$

The resulting KKT conditions are

$$\begin{aligned} \nabla \mathcal{L} = & - \begin{bmatrix} \tilde{\mathbf{h}}_1^H \mathbf{J}^{-2} \tilde{\mathbf{h}}_1 \\ \vdots \\ \tilde{\mathbf{h}}_L^H \mathbf{J}^{-2} \tilde{\mathbf{h}}_L \end{bmatrix} + \mu_{\text{sum}} \mathbf{1}_L - \sum_{l=1}^L \mu_l \mathbf{e}_l = \mathbf{0}_L \\ & \sum_{l=1}^L q_l \leq P_{\text{max}}, \quad q_l \geq 0 \\ & \mu_{\text{sum}} \geq 0, \mu_l \geq 0 \\ & \mu_{\text{sum}} \left(\sum_{l=1}^L q_l - P_{\text{max}} \right) = 0, \quad \mu_l q_l = 0. \end{aligned} \quad (9)$$

Here, $\mathbf{0}_L$ is the length- L all-zeroes vector, and \mathbf{e}_l is the standard basis vector with a single one in the l^{th} position and zeroes elsewhere. The gradient in the stationarity condition follows from the inverse differential identity $\partial \mathbf{J}^{-1} = -\mathbf{J}^{-1} (\partial \mathbf{J}) \mathbf{J}^{-1}$ [7] and the linearity of the trace operator; thus,

$$\begin{aligned} \frac{\partial \text{tr} [\mathbf{J}^{-1}]}{\partial q_l} &= \text{tr} \left[-\mathbf{J}^{-1} \frac{\partial \mathbf{J}}{\partial q_l} \mathbf{J}^{-1} \right] = -\text{tr} \left[\mathbf{J}^{-1} \tilde{\mathbf{h}}_l \tilde{\mathbf{h}}_l^H \mathbf{J}^{-1} \right] \\ &= -\tilde{\mathbf{h}}_l^H \mathbf{J}^{-2} \tilde{\mathbf{h}}_l. \end{aligned} \quad (10)$$

B. Conditions for Equality under Optimal Power Allocation

Having solved (4) for an arbitrary set of virtual uplink precoders \mathbf{v}_l , we then find the MMSE receive beamformers $\tilde{\mathbf{u}}_l = \mathbf{J}^{-1} \tilde{\mathbf{h}}_l \sqrt{\tilde{q}_l}$. In order to preserve the sum power constraint after transformation to the downlink, we normalize these beamformers as $\mathbf{u}_l = \tilde{\mathbf{u}}_l / \|\tilde{\mathbf{u}}_l\|$. With this selected set of $(\mathbf{v}_l, \tilde{q}_l, \mathbf{u}_l)$, we can find the associated virtual uplink SINRs γ_l^{UL} and set these as the (achievable) target SINRs γ_l in (6). Thus, a feasible power allocation is possible in the downlink to achieve the same MSE for each data stream.

In the case where the optimal power allocation results in one or more *inactive streams* $\mathcal{S}_I = \{l \in (1, \dots, L) \mid q_l = 0\}$, the power transformation algorithm in (5) fails due to inversion of the singular matrix \mathbf{D} . However, the same MSEs can be achieved for these inactive streams in the downlink by setting $p_l = 0$ for $l \in \mathcal{S}_I$. The power allocation \mathbf{p} for the set of *active streams* $\mathcal{S}_A = \{l \in (1, \dots, L) \mid q_l > 0\}$ can then be found by following the specified procedure after deleting the rows and columns from \mathbf{D} and Ψ corresponding to the inactive streams.

The coupling matrix Ψ is a real matrix whose off-diagonal entries $[\Psi]_{ij}$ contains squared magnitudes of the end-to-end channel gains from transmitted symbol x_j to the decoded symbol \hat{x}_i . We observe that transpose symmetry $\Psi = \Psi^T$ is satisfied when

$$\begin{aligned} \frac{\tilde{\mathbf{h}}_i^H \tilde{\mathbf{u}}_j}{\|\tilde{\mathbf{u}}_j\|} &= \frac{\tilde{\mathbf{u}}_i^H \tilde{\mathbf{h}}_j}{\|\tilde{\mathbf{u}}_i\|} \\ \frac{\tilde{\mathbf{h}}_i^H \mathbf{J}^{-1} \tilde{\mathbf{h}}_j \sqrt{\tilde{q}_j}}{\sqrt{\tilde{q}_j \tilde{\mathbf{h}}_j^H \mathbf{J}^{-2} \tilde{\mathbf{h}}_j}} &= \frac{\sqrt{\tilde{q}_i} \tilde{\mathbf{h}}_i^H \mathbf{J}^{-1} \tilde{\mathbf{h}}_j}{\sqrt{\tilde{q}_i \tilde{\mathbf{h}}_i^H \mathbf{J}^{-2} \tilde{\mathbf{h}}_i}} \end{aligned} \quad (11)$$

The power terms \tilde{q}_i and \tilde{q}_j are cancelled out, and numerators are equal; thus, an equivalent expression for the sufficient condition for $\mathbf{p} = \mathbf{q}$ is

$$\tilde{\mathbf{h}}_i^H \mathbf{J}^{-2} \tilde{\mathbf{h}}_i = \tilde{\mathbf{h}}_j^H \mathbf{J}^{-2} \tilde{\mathbf{h}}_j \quad \forall i, j \in \mathcal{S}_A. \quad (12)$$

We rewrite the individual terms in the first KKT condition as $\tilde{\mathbf{h}}_l^H \mathbf{J}^{-2} \tilde{\mathbf{h}}_l = (\mu_{\text{sum}} - \mu_l)$. Due to the complementary slackness condition $\mu_l q_l = 0$ in (9), the dual variables μ_l are zero for all active streams $l \in \mathcal{S}_A$ with $q_l > 0$. Thus, it follows that

$$\tilde{\mathbf{h}}_l^H \mathbf{J}^{-2} \tilde{\mathbf{h}}_l = \mu_{\text{sum}} \quad \forall l \in \mathcal{S}_A; \quad (13)$$

that is, (12) is satisfied, $\Psi = \Psi^T$, and the downlink and virtual uplink power allocations \mathbf{p} and \mathbf{q} that satisfy the same set of per-stream MSEs and equal sum-MSE are identical.

C. Discussion

The equality result presented in Section III was shown to apply for arbitrary \mathbf{v}_l , as long as the optimum power allocation and MMSE receivers are used. This result implies that the virtual uplink to downlink transformation stage can be omitted from algorithms using both iterative and joint designs [1]–[4], thus allowing for simplified implementations. Since the virtual uplink solutions are solved using convex programming, eliminating the power transformation step (with its associated matrix inversion) may represent an appreciable decrease in computational complexity.

IV. CONCLUSIONS

In this letter, we have proven that the optimum power allocations for the downlink and virtual uplink are identical when minimizing the sum-MSE under a sum power constraint. With this proof, we extend the known results in a well studied problem, and decrease the computational complexity required to implement existing algorithms.

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