

Linear Processing and Sum Throughput in the Multiuser MIMO Downlink

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Abstract—We consider linear precoding and decoding in the downlink of a multiuser multiple-input, multiple-output (MIMO) system, wherein each user may receive more than one data stream. We propose several mean squared error (MSE) based criteria for joint transmit-receive optimization and establish a series of relationships linking these criteria to the signal-to-interference-plus-noise ratios of individual data streams and the information theoretic channel capacity under linear minimum MSE decoding. In particular, we show that achieving the maximum sum throughput is equivalent to minimizing the product of MSE matrix determinants (PDetMSE). Since the PDetMSE minimization problem does not admit a computationally efficient solution, a simplified scalar version of the problem is considered that minimizes the product of mean squared errors (PMSE). An iterative algorithm is proposed to solve the PMSE problem, and is shown to provide near-optimal performance with greatly reduced computational complexity. Our simulations compare the achievable sum rates under linear precoding strategies to the sum capacity for the broadcast channel.

Index Terms—Adaptive arrays, adaptive modulation, diversity methods, information rates, MIMO systems, nonlinear programming, optimization methods, least mean square methods, space division multiplexing.

I. INTRODUCTION

THE benefits of using multiple antennas for wireless communication systems are well known. When antenna arrays are present at the transmitter and/or receiver, multiple-input multiple-output (MIMO) techniques can utilize the spatial dimension to yield improved reliability, increased data rates, and the spatial separation of users. In this paper, the methods we propose will focus on exploiting all of these features, with the goal of maximizing the sum data rate achieved in the MIMO multiuser downlink.

The optimal strategy for maximizing sum rate in the multiuser MIMO downlink, also known as the broadcast channel (BC), was first proposed in [1]; the authors prove that Costa's dirty paper coding (DPC) strategy [2] is sum capacity achieving for a pair of single-antenna users. The sum-rate optimality of DPC was generalized to an arbitrary number of multi-antenna receivers using the notions of game theory [3] and

uplink-downlink duality [4], [5]; this duality is employed in [6], [7] to derive iterative solutions that find the sum capacity. DPC has been shown to be the optimal precoding strategy not only for sum capacity, but also for the entire capacity region in the BC [8]. Unfortunately, finding a practical realization of the DPC precoding strategy has proven to be a difficult problem. Existing solutions, which are largely based on Tomlinson-Harashima precoding (THP) [9]–[12], incur high complexity due to their nonlinear nature and the combinatorial problem of user order selection. THP-based schemes also suffer from rate loss when compared to the sum capacity due to modulo and shaping losses.

Linear precoding provides an alternative approach for transmission in the MIMO downlink, trading off a reduction in precoder complexity for suboptimal performance. Orthogonalization based schemes use zero forcing (ZF) and block diagonalization (BD) to transform the multiuser downlink into parallel single-user systems [13], [14]. A waterfilling power allocation can then be used to allocate powers to each of the users [15]. The simplicity of these approaches comes at the expense of an antenna constraint requiring at least as many transmit antennas as the total number of receive antennas. These schemes, therefore, restrict the possibility of gains from additional receiver antennas. The constraint is relaxed under successive zero forcing [16], which requires only partial orthogonality but incurs higher complexity in finding an optimal user ordering. Coordinated beamforming [17] and generalized orthogonalization [18] are able to avoid the antenna constraint via iterative optimization of transmit and receive beamformers.

It is also possible to improve the sum rate achieved with ZF and BD by including user or antenna selection in the precoder design. The sum-rate maximizing ZF precoder can be found by comparing precoders for all possible subsets of available receive antennas [1]; however, this strategy incurs exponential complexity on the order of the total number of receive antennas. Greedy and suboptimal strategies for user selection [19]–[22] may also be applied with lower computational cost. However, user selection is outside the scope of this paper; our goal here is to focus on the rates achievable under linear precoding. While all of these schemes possess lower complexity than the THP based methods, the use of orthogonalization results in suboptimal performance due to noise enhancement. In this paper, we consider the optimal formulation for sum rate maximization under linear precoding.

Much of the existing literature on linear precoding for multiuser MIMO systems focuses on minimizing the sum of mean squared errors (SMSE) between the transmitted and received signals under a sum power constraint [23]–[28]. An

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important recurring theme in most of these papers is the use of an uplink-downlink duality for both MSE and signal-to-interference-plus-noise ratio (SINR) introduced in [24] for the single receive antenna case and extended to the MIMO case in [26], [27]. These MSE and SINR dualities are equally applicable to sum rate maximization.

Linear precoding approaches to sum rate maximization have been proposed for both single-antenna receivers [29], [30] and for multiple antenna receivers [31]–[33]. In [29], the authors suggest an iterative method for direct optimization of the sum rate, while [30] and [31] exploit the SINR uplink-downlink duality of [24], [26], [27]. In [32] and [33], two similar algorithms were independently proposed to minimize the product of the mean squared errors (PMSE) in the multiuser MIMO downlink; these papers showed that the PMSE minimization problem is equivalent to the direct sum rate maximization proposed in [29]–[31]. The work of [33] was motivated by the equivalence relationship developed between the single user minimum MSE (MMSE) and mutual information in [34]. Each of the approaches in [29]–[33] yields a suboptimal solution, as the resulting solutions converge only to a local optimum, if at all.

Given this prior work in linear precoding, an important motivation for this paper is to determine the performance upper bound achievable under linear precoding and to evaluate how closely PMSE minimization comes to approaching this upper bound. In the single-user multicarrier case, minimizing the PMSE is equivalent to minimizing the determinant of the MSE matrix and thus is also equivalent to maximizing the mutual information [35]. This equivalence does not apply to the multiuser scenario. In this paper, we investigate the relationship between the MSE-matrix determinants, the mutual information, and the maximum achievable sum rate under linear precoding in the multiuser MIMO downlink, resulting in an optimization problem based on minimizing the product of the determinants of all users' MSE matrices (PDetMSE). Furthermore, we underline the differences between the joint (multi-stream) optimization that arises from the PDetMSE approach and the scalar (per-stream) PMSE-based solution. While chronologically, the PMSE approach was developed before the PDetMSE formulation, we present PMSE in this paper as a lower complexity approximation of the PDetMSE formulation.

The main contributions of this paper are:

- Deriving the maximum achievable information rates for both joint and scalar processing under linear precoding and formulating the joint (PDetMSE) and scalar processing (PMSE) based sum rate maximization problems using MSE expressions.
- Proposing solutions to these optimization problems based on uplink-downlink duality, and addressing several issues regarding algorithm implementation.
- Analyzing the performance of our proposed schemes in comparison to the DPC sum capacity and to orthogonalization based approaches. We demonstrate that a performance improvement is made in narrowing the gap to capacity at practical values of transmit SNR, and show that the PDetMSE approach provides the best performance of all proposed schemes.

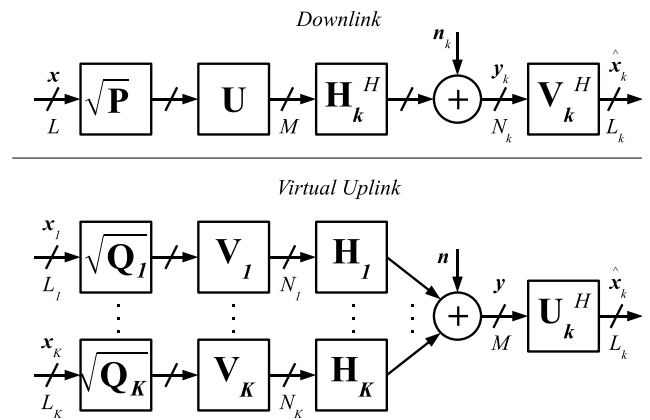


Fig. 1. Processing for user k in downlink and virtual uplink.

The remainder of this paper is organized as follows. Section II describes the system model used and states the assumptions made. Section III derives the performance upper bound for the achievable sum rate under linear precoding, and develops the use of the product of MSE matrix determinants as the optimization criterion for joint processing. Section IV investigates a suboptimal framework based on the product of mean squared errors and proposes a computationally feasible scheme for implementation. Results of simulations testing the effectiveness of the proposed approaches are presented in Section V. Finally, we draw our conclusions in Section VI.

Notation: Lower case italics, e.g., x , represent scalars while lower case boldface type is used for vectors (e.g., \mathbf{x}). Upper case italics, e.g., N , are used for constants and upper case boldface represents matrices, e.g., \mathbf{X} . Entries in vectors and matrices are denoted as $[\mathbf{x}]_i$ and $[\mathbf{X}]_{i,j}$ respectively. The superscripts T and H denote the transpose and Hermitian operators. $\mathbb{E}[\cdot]$ represents the statistical expectation operator while \mathbf{I}_N is the $N \times N$ identity matrix. $\text{tr}[\cdot]$ and $\det(\cdot)$ are the trace and determinant operators. $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_2$ denote the 1-norm (sum of entries) and Euclidean norm. $\text{diag}(\mathbf{x})$ represents the diagonal matrix formed using the entries in vector \mathbf{x} , and $\text{diag}[\mathbf{X}_1, \dots, \mathbf{X}_k]$ is the block diagonal concatenation of matrices $\mathbf{X}_1, \dots, \mathbf{X}_k$. $\mathbf{A} \succ \mathbf{0}$ and $\mathbf{B} \succeq \mathbf{0}$ indicate that \mathbf{A} and \mathbf{B} are positive definite and positive semidefinite matrices, respectively. $\hat{e}_{\max}(\mathbf{A}, \mathbf{B})$ is the unit Euclidean norm eigenvector \mathbf{x} corresponding to the largest eigenvalue λ in the generalized eigenproblem $\mathbf{A}\mathbf{x} = \lambda\mathbf{B}\mathbf{x}$. Finally, $\mathcal{CN}(m, \sigma^2)$ denotes the complex Gaussian probability distribution with mean m and variance σ^2 .

II. SYSTEM MODEL WITH LINEAR PRECODING

The system under consideration, illustrated in Fig. 1, comprises a base station with M antennas transmitting to K decentralized users over flat wireless channels. User k is equipped with N_k antennas and receives L_k data streams from the base station. Thus, we have M transmit antennas transmitting a total of $L = \sum_{k=1}^K L_k$ symbols to K users, who, together, have a total of $N = \sum_{k=1}^K N_k$ receive antennas. The data symbols for user k are collected in the data vector $\mathbf{x}_k = [x_{k1}, x_{k2}, \dots, x_{kL_k}]^T$ and the overall data vector is

$\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_K^T]^T$. We assume that the modulated data symbols \mathbf{x} are independent with unit average energy ($\mathbb{E}[\mathbf{x}\mathbf{x}^H] = \mathbf{I}_L$). User k 's data streams are processed by the $M \times L_k$ transmit filter $\mathbf{U}_k = [\mathbf{u}_{k1}, \dots, \mathbf{u}_{kL_k}]$ before being transmitted over the M antennas; \mathbf{u}_{kj} is the precoder for stream j of user k , and has unit power $\|\mathbf{u}_{kj}\|_2 = 1$. Together, these individual precoders form the $M \times L$ global transmitter precoder matrix $\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_K]$. Let p_{kj} be the power allocated to stream j of user k and the downlink transmit power vector for user k be $\mathbf{p}_k = [p_{k1}, p_{k2}, \dots, p_{kL_k}]^T$, with $\mathbf{p} = [\mathbf{p}_1^T, \dots, \mathbf{p}_K^T]^T$. Define $\mathbf{P}_k = \text{diag}\{\mathbf{p}_k\}$ and $\mathbf{P} = \text{diag}\{\mathbf{p}\}$. The channel between the transmitter and user k is represented by the $N_k \times M$ matrix \mathbf{H}_k^H . The overall $N \times M$ channel matrix is \mathbf{H}^H , with $\mathbf{H} = [\mathbf{H}_1, \mathbf{H}_2, \dots, \mathbf{H}_K]$. The transmitter is assumed to know the channel perfectly.

Based on this model, user k receives a length- N_k vector

$$\mathbf{y}_k = \mathbf{H}_k^H \mathbf{U} \sqrt{\mathbf{P}} \mathbf{x} + \mathbf{n}_k,$$

where \mathbf{n}_k consists of the additive white Gaussian noise (AWGN) at the user's receive antennas with i.i.d. entries $[\mathbf{n}_k]_i \sim \mathcal{CN}(0, \sigma^2)$; that is, $\mathbb{E}[\mathbf{n}_k \mathbf{n}_k^H] = \sigma^2 \mathbf{I}_{N_k}$. To estimate its L_k symbols \mathbf{x}_k , user k processes \mathbf{y}_k with its $L_k \times N_k$ decoder matrix \mathbf{V}_k^H resulting in

$$\hat{\mathbf{x}}_k^{DL} = \mathbf{V}_k^H \mathbf{H}_k^H \mathbf{U} \sqrt{\mathbf{P}} \mathbf{x} + \mathbf{V}_k^H \mathbf{n}_k,$$

where the superscript DL indicates the downlink. The global receive filter \mathbf{V}^H is a block diagonal matrix of dimension $L \times N$, $\mathbf{V} = \text{diag}[\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_K]$, where each $\mathbf{V}_k = [\mathbf{v}_{k1}, \dots, \mathbf{v}_{kL_k}]$. The MSE matrix for user k in the downlink under these general precoder and decoder matrices can be written as

$$\begin{aligned} \mathbf{E}_k^{DL} &= \mathbb{E}[(\hat{\mathbf{x}}_k - \mathbf{x}_k)(\hat{\mathbf{x}}_k - \mathbf{x}_k)^H] \\ &= \mathbf{V}_k^H \mathbf{H}_k^H \mathbf{U} \mathbf{P} \mathbf{U}^H \mathbf{H}_k \mathbf{V}_k + \sigma^2 \mathbf{V}_k^H \mathbf{V}_k \\ &\quad - \mathbf{V}_k^H \mathbf{H}_k^H \mathbf{U} \sqrt{\mathbf{P}_k} - \sqrt{\mathbf{P}_k} \mathbf{U}^H \mathbf{H}_k \mathbf{V}_k + \mathbf{I}_{L_k}. \end{aligned} \quad (1)$$

We will make use of the dual virtual uplink, also illustrated in Fig. 1, with the same channels between users and base station. In the uplink, user k transmits L_k data streams. Let the uplink transmit power vector for user k be $\mathbf{q}_k = [q_{k1}, q_{k2}, \dots, q_{kL_k}]^T$, with $\mathbf{q} = [\mathbf{q}_1^T, \dots, \mathbf{q}_K^T]^T$, and define $\mathbf{Q}_k = \text{diag}\{\mathbf{q}_k\}$ and $\mathbf{Q} = \text{diag}\{\mathbf{q}\}$. The transmit and receive filters for user k become \mathbf{V}_k and \mathbf{U}_k^H respectively. As in the downlink, the precoder for the virtual uplink contains columns with unit norm; that is, $\|\mathbf{v}_{kj}\|_2 = 1$. The received vector at the base station and the estimated symbol vector for user k are

$$\begin{aligned} \mathbf{y} &= \sum_{i=1}^K \mathbf{H}_i \mathbf{V}_i \sqrt{\mathbf{Q}_i} \mathbf{x}_i + \mathbf{n}, \\ \hat{\mathbf{x}}_k^{UL} &= \sum_{i=1}^K \mathbf{U}_k^H \mathbf{H}_i \mathbf{V}_i \sqrt{\mathbf{Q}_i} \mathbf{x}_i + \mathbf{U}_k^H \mathbf{n}. \end{aligned}$$

The noise term, \mathbf{n} , is again AWGN with $\mathbb{E}[\mathbf{n}\mathbf{n}^H] = \sigma^2 \mathbf{I}_M$.

We define a useful virtual uplink receive covariance matrix

as

$$\begin{aligned} \mathbf{J} &= \mathbb{E}[\mathbf{y}\mathbf{y}^H] = \sum_{k=1}^K \mathbf{H}_k \mathbf{V}_k \mathbf{Q}_k \mathbf{V}_k^H \mathbf{H}_k^H + \sigma^2 \mathbf{I}_M \\ &= \mathbf{H} \mathbf{V} \mathbf{Q} \mathbf{V}^H \mathbf{H}^H + \sigma^2 \mathbf{I}_M. \end{aligned}$$

The global MSE matrix for all users in the virtual uplink can then be expressed as

$$\begin{aligned} \mathbf{E}^{UL} &= \mathbb{E}[(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^H] \\ &= \mathbf{U}^H \mathbf{J} \mathbf{U} - \mathbf{U}^H \mathbf{H} \mathbf{V} \sqrt{\mathbf{Q}} - \sqrt{\mathbf{Q}} \mathbf{V}^H \mathbf{H} \mathbf{U} + \mathbf{I}_L. \end{aligned} \quad (2)$$

III. LINEAR PRECODING AND SUM RATE MAXIMIZATION

In this section, we formulate the sum rate maximization problem under linear precoding in the broadcast channel. We begin by introducing the information theoretic DPC upper bound, and then derive the performance upper bound achievable under linear precoding. We then derive an equivalent formulation in terms of MSE expressions, and propose the PDetMSE based scheme for achieving this optimal sum rate performance under linear precoding.

A. Sum Capacity and Dirty Paper Coding

Information theoretic approaches characterize the sum capacity of the multiuser MIMO downlink by solving the sum capacity of the equivalent uplink multiple access channel (MAC) and applying a duality result [4], [5]. The BC sum capacity can thus be expressed as

$$\begin{aligned} R_{\text{sum}} &= \max_{\mathbf{\Sigma}_k} \log \det \left(\mathbf{I} + \frac{1}{\sigma^2} \sum_{k=1}^K \mathbf{H}_k \mathbf{\Sigma}_k \mathbf{H}_k^H \right) \\ \text{s.t.} \quad &\mathbf{\Sigma}_k \succeq \mathbf{0}, \quad k = 1, \dots, K \\ &\sum_{k=1}^K \text{tr}[\mathbf{\Sigma}_k] \leq P_{\text{max}}, \end{aligned}$$

where $\mathbf{\Sigma}_k$ is the uplink transmit covariance matrix for mobile user k , and P_{max} is the maximum sum power over all users. Note that this optimization problem is concave in $\mathbf{\Sigma}_k$, and is hence relatively easy to solve. This result does not translate to linear precoding.

B. Achievable Sum Rate under Linear Precoding

The achievable rate for a single user MIMO channel is $\log(\det(\mathbf{K}_x + \mathbf{K}_z)/\det(\mathbf{K}_z))$ (where \mathbf{K}_x is the received signal covariance and \mathbf{K}_z is the noise covariance) [36]. Under single-user decoding, multi-user interference is treated as noise, and user k can achieve rate R_k in the downlink using transmit covariance $\mathbf{\Sigma}_k$:

$$R_k = \log \frac{\det \left(\sum_{j=1}^K \mathbf{H}_k^H \mathbf{\Sigma}_j \mathbf{H}_k + \sigma^2 \mathbf{I} \right)}{\det \left(\sum_{j \neq k} \mathbf{H}_k^H \mathbf{\Sigma}_j \mathbf{H}_k + \sigma^2 \mathbf{I} \right)}.$$

Under the system model described in Section II, user k transmits with covariance matrix $\mathbf{\Sigma}_k = \mathbf{U}_k \mathbf{P}_k \mathbf{U}_k^H$. The

achievable rate for user k under linear precoding is therefore

$$\begin{aligned} R_k^{\text{LP}} &= \log \frac{\det \left(\sum_{j=1}^K \mathbf{H}_k^H \mathbf{U}_j \mathbf{P}_j \mathbf{U}_j^H \mathbf{H}_k + \sigma^2 \mathbf{I} \right)}{\det \left(\sum_{j \neq k} \mathbf{H}_k^H \mathbf{U}_j \mathbf{P}_j \mathbf{U}_j^H \mathbf{H}_k + \sigma^2 \mathbf{I} \right)} \\ &= \log \frac{\det \mathbf{J}_k}{\det \mathbf{R}_{N+I,k}}, \end{aligned} \quad (3)$$

where $\mathbf{J}_k = \mathbf{H}_k^H \mathbf{U} \mathbf{P} \mathbf{U}^H \mathbf{H}_k + \sigma^2 \mathbf{I}$ and $\mathbf{R}_{N+I,k} = \mathbf{J}_k - \mathbf{H}_k^H \mathbf{U}_k \mathbf{P}_k \mathbf{U}_k^H \mathbf{H}_k$ are the received signal covariance matrix and the noise-plus-interference covariance matrix at user k , respectively.

The rate maximization problem with a sum power constraint under linear precoding can then be formulated as

$$\begin{aligned} (\mathbf{U}, \mathbf{P}) &= \arg \max_{\mathbf{U}, \mathbf{P}} \sum_{k=1}^K \log \frac{\det \mathbf{J}_k}{\det \mathbf{R}_{N+I,k}} \\ \text{s.t. } &\|\mathbf{u}_{kj}\|_2 = 1, \quad k = 1, \dots, K, \quad j = 1, \dots, L_k \\ &p_{kj} \geq 0, \quad k = 1, \dots, K, \quad j = 1, \dots, L_k \\ &\|\mathbf{P}\|_1 = \sum_{k=1}^K \sum_{j=1}^{L_k} p_{kj} \leq P_{\max}. \end{aligned} \quad (4)$$

C. MSE Formulation: Product of MSE Matrix Determinants

In this section, we show that an MSE-based formulation using joint processing of all streams (rather than treating each user's own data streams as interference) leads to an equivalent optimal formulation of the rate maximization problem under linear processing. We develop this relationship by using the MSE matrix determinants.

First, consider the linear MMSE decoder for user k , \mathbf{V}_k ,

$$\begin{aligned} \mathbf{V}_k &= \left(\mathbf{H}_k^H \mathbf{U} \mathbf{P} \mathbf{U}^H \mathbf{H}_k + \sigma^2 \mathbf{I} \right)^{-1} \mathbf{H}_k^H \mathbf{U}_k \sqrt{\mathbf{P}_k} \\ &= \mathbf{J}_k^{-1} \mathbf{H}_k^H \mathbf{U}_k \sqrt{\mathbf{P}_k}. \end{aligned} \quad (5)$$

When using this matrix as the receiver in (1), the downlink MSE matrix for user k in can be simplified as

$$\mathbf{E}_k^{DL} = \mathbf{I}_{L_k} - \sqrt{\mathbf{P}_k} \mathbf{U}_k^H \mathbf{H}_k \mathbf{J}_k^{-1} \mathbf{H}_k^H \mathbf{U}_k \sqrt{\mathbf{P}_k}. \quad (6)$$

Consider the following optimization problem which minimizes the product of the determinants of the downlink MSE matrices under a sum power constraint:

$$\begin{aligned} (\mathbf{U}, \mathbf{P}) &= \arg \min_{\mathbf{U}, \mathbf{P}} \prod_{k=1}^K \det \mathbf{E}_k^{DL} \\ \text{s.t. } &\|\mathbf{u}_{kj}\|_2 = 1, \quad k = 1, \dots, K, \quad j = 1, \dots, L_k \\ &p_{kj} \geq 0, \quad k = 1, \dots, K, \quad j = 1, \dots, L_k \\ &\|\mathbf{P}\|_1 = \sum_{k=1}^K \sum_{j=1}^{L_k} p_{kj} \leq P_{\max}. \end{aligned} \quad (7)$$

Theorem 1: Under linear MMSE decoding at the base station, the sum rate maximization problem in (4) and the PDetMSE minimization problem in (7) are equivalent.

Proof: The determinant of the downlink MSE matrix can be written as

$$\begin{aligned} \det \mathbf{E}_k^{DL} &= \det \left(\mathbf{I}_{L_k} - \mathbf{H}_k^H \mathbf{U}_k \mathbf{P}_k \mathbf{U}_k^H \mathbf{H}_k \mathbf{J}_k^{-1} \right) \\ &= \det \left[\left(\mathbf{J}_k - \mathbf{H}_k^H \mathbf{U}_k \mathbf{P}_k \mathbf{U}_k^H \mathbf{H}_k \right) \mathbf{J}_k^{-1} \right] \\ &= \det \left[\mathbf{R}_{N+I,k} \mathbf{J}_k^{-1} \right] \\ &= \frac{\det \mathbf{R}_{N+I,k}}{\det \mathbf{J}_k}, \end{aligned} \quad (8)$$

where (8) follows from (6) since $\det(\mathbf{I} + \mathbf{A}\mathbf{B}) = \det(\mathbf{I} + \mathbf{B}\mathbf{A})$ when \mathbf{A} and \mathbf{B} have appropriate dimensions. We then see the relationship to (3),

$$\begin{aligned} \log \det \mathbf{E}_k^{DL} &= -\log \frac{\det \mathbf{J}_k}{\det \mathbf{R}_{N+I,k}} \\ &= -R_k^{\text{LP}}. \end{aligned}$$

With this result, we can see that under MMSE reception using \mathbf{V}_k as defined in (5), minimizing the determinant of the MSE matrix \mathbf{E}_k^{DL} is equivalent to maximizing the achievable rate for user k . It follows that minimizing the product of MSE matrix determinants over all users is equivalent to sum rate maximization,

$$\begin{aligned} \min \prod_{k=1}^K \det \mathbf{E}_k^{DL} &\equiv \min \sum_{k=1}^K \log \det \mathbf{E}_k^{DL} \\ &\equiv \max \sum_{k=1}^K R_k^{\text{LP}}. \end{aligned} \quad (9)$$

where (9) holds since since $\log(\cdot)$ is a monotonically increasing function of its argument. ■

Note that this new result represents an upper bound on the sum rate on all linear precoding schemes in the broadcast channel.

The covariance matrices \mathbf{J}_k and $\mathbf{R}_{N+I,k}$ in the MSE matrix \mathbf{E}_k are each functions of all precoder and power allocation matrices. Thus, the sum rates R_k for each user k (and the sum rate for all users) are coupled across users. As such, finding \mathbf{U} and \mathbf{P} jointly or finding only the power allocation \mathbf{P} for a fixed \mathbf{U} are both non-convex problems and are just as difficult to solve as the rate maximization problem.

In the sum capacity and SMSE problems, the problem of non-convexity is addressed by solving a convex virtual uplink formulation and applying a duality-based transformation. Unfortunately, the sum rate expression under linear precoding in the virtual uplink is nearly identical to that derived above for the downlink, and does not admit a cancellation or grouping of terms to decouple the problem across users.

Direct solution of the non-convex downlink problem for minimizing the product of MSE matrix determinants requires finding a complex $M \times L$ precoder matrix. We consider the application of sequential quadratic programming (SQP) [37] to solve this problem. SQP solves successive approximations of a constrained optimization problem and is guaranteed to converge to the optimum value for convex problems; however, in the case of this non-convex optimization problem, SQP can only guarantee convergence to a local minimum.

This computationally intensive approach is the only available option in the absence of a convex virtual uplink formulation. Moreover, the numerical techniques used for solving

nonlinear problems do not guarantee convergence to the global minimum. This is clearly not a desirable method for finding a practical precoder, especially when one of our major motivations for using linear precoding is reducing transmitter complexity. We do not suggest that this method be practically implemented; rather, we use it to illustrate the difference in performance between the solutions to the optimal PDetMSE formulation and the more practical PMSE algorithm that we propose in the following section.

IV. SCALAR PROCESSING AND THE PRODUCT OF MEAN SQUARED ERRORS

Given the complexity of the PDetMSE solution, we consider PMSE minimization as a suboptimal (but likely feasible) approximation to rate maximization in the multiuser MIMO downlink. In [35], the single-user rate maximization problem using linear precoding is solved by minimizing the determinant of the MSE matrix. This solution is equivalent to minimizing the product of individual stream MSEs because the problem is scalarized by diagonalization of both the channel and MSE matrices. It was recently demonstrated in [38] that the MSE matrices can also be diagonalized in the multiuser case by applying unitary transformations to the precoder and decoders; however, in the absence of orthogonalizing precoders (e.g., BD or ZF), minimization of the PMSE yields a different solution from minimizing the PDetMSE.

The PMSE approach, based on scalar processing of the individual stream MSEs for each user, follows from the treatment of the optimization problems in [26], [27], where non-convex problems in the downlink are transformed to convex problems in the dual uplink. With this motivation in mind, we consider formulating the scalar optimization problem directly in the virtual uplink, and transforming the resulting solution to the downlink using the uplink-downlink MSE duality in [26], [27].

A. Achievable Sum Rate using Scalar Processing

In the scalarized version of the rate maximization problem, the user's own data streams ($l \neq j$) are considered as self-interference in addition to the multiuser interference. The achievable rate for user k 's substream j can thus be expressed as

$$R_{k,j}^{\text{LP}} = \log(1 + \gamma_{k,j}^{UL}),$$

where

$$\gamma_{k,j}^{UL} = \frac{\mathbf{u}_{k,j}^H \mathbf{H}_k \mathbf{v}_{k,j} q_{k,j} \mathbf{v}_{k,j}^H \mathbf{H}_k^H \mathbf{u}_{k,j}}{\mathbf{u}_{k,j}^H \mathbf{J}_{k,j} \mathbf{u}_{k,j}} \quad (10)$$

is the SINR and $\mathbf{J}_{k,j} = \mathbf{J} - \mathbf{H}_k \mathbf{v}_{k,j} q_{k,j} \mathbf{v}_{k,j}^H \mathbf{H}_k^H$ is the virtual uplink interference-plus-noise covariance matrix for stream j of user k .

The scalar rate maximization problem with a sum power constraint under linear precoding can thus be written as

$$\begin{aligned} (\mathbf{V}, \mathbf{Q}) &= \arg \max_{\mathbf{V}, \mathbf{Q}} \sum_{k=1}^K \sum_{j=1}^{L_k} \log(1 + \gamma_{k,j}^{UL}) \\ \text{s.t. } &\|\mathbf{v}_{k,j}\|_2 = 1, \quad k = 1, \dots, K, \quad j = 1, \dots, L_k \\ &q_{k,j} \geq 0, \quad k = 1, \dots, K, \quad j = 1, \dots, L_k \\ &\|\mathbf{q}\|_1 = \sum_{k=1}^K \sum_{j=1}^{L_k} q_{k,j} \leq P_{\max}. \end{aligned} \quad (11)$$

B. MSE Formulation: Product of Mean Squared Errors

With this scalar processing rate maximization problem in mind, we consider the MSE-equivalent formulation. We begin by finding the optimum linear receiver, and can see from (10) that $\mathbf{u}_{k,j}$ does not depend on any other columns of \mathbf{U} . Furthermore, it is the solution to the generalized eigenproblem

$$\mathbf{u}_{k,j}^{\text{opt}} = \hat{e}_{\max}(\mathbf{H}_k \mathbf{v}_{k,j} q_{k,j} \mathbf{v}_{k,j}^H \mathbf{H}_k^H, \mathbf{J}_{k,j}).$$

Within a normalizing factor, this solution is equivalent to the MMSE receiver,

$$\mathbf{u}_{k,j}^{\text{opt}} = \mathbf{J}^{-1} \mathbf{H}_k \mathbf{v}_{k,j} \sqrt{q_{k,j}}. \quad (12)$$

When the MMSE receiver in (12) is used, the virtual uplink MSE matrix (2) reduces to

$$\mathbf{E}^{UL} = \mathbf{I}_L - \sqrt{\mathbf{Q}} \mathbf{V}^H \mathbf{H}^H \mathbf{J}^{-1} \mathbf{H} \mathbf{V} \sqrt{\mathbf{Q}}.$$

Thus, the mean squared error for user k 's j^{th} stream is entry j in block k of \mathbf{E}^{UL} ,

$$\epsilon_{k,j}^{UL} = 1 - q_{k,j} \mathbf{v}_{k,j}^H \mathbf{H}_k^H \mathbf{J}^{-1} \mathbf{H}_k \mathbf{v}_{k,j}.$$

Now consider another optimization problem, minimizing the product of mean squared errors (PMSE) under a sum power constraint,

$$\begin{aligned} (\mathbf{V}, \mathbf{Q}) &= \arg \min_{\mathbf{V}, \mathbf{Q}} \prod_{k=1}^K \prod_{j=1}^{L_k} \epsilon_{k,j}^{UL} \\ \text{s.t. } &\|\mathbf{v}_{k,j}\|_2 = 1, \quad k = 1, \dots, K, \quad j = 1, \dots, L_k \\ &q_{k,j} \geq 0, \quad k = 1, \dots, K, \quad j = 1, \dots, L_k \\ &\|\mathbf{q}\|_1 = \sum_{k=1}^K \sum_{j=1}^{L_k} q_{k,j} \leq P_{\max}. \end{aligned} \quad (13)$$

Theorem 2: Under linear MMSE decoding at the base station, the optimization problems defined by (11) and (13) are equivalent.

Proof: Using (10), we can rewrite $1 + \gamma_{k,j}^{UL}$ as

$$1 + \gamma_{k,j}^{UL} = \frac{\mathbf{u}_{k,j}^H \mathbf{J}_{k,j} \mathbf{u}_{k,j}}{\mathbf{u}_{k,j}^H \mathbf{J}_{k,j} \mathbf{u}_{k,j} - \mathbf{u}_{k,j}^H \mathbf{H}_k \mathbf{v}_{k,j} q_{k,j} \mathbf{v}_{k,j}^H \mathbf{H}_k^H \mathbf{u}_{k,j}}.$$

It follows that by using the MMSE receiver from (12),

$$\begin{aligned} \frac{1}{1 + \gamma_{k,j}^{UL}} &= 1 - \frac{\mathbf{u}_{k,j}^H \mathbf{H}_k \mathbf{v}_{k,j} q_{k,j} \mathbf{v}_{k,j}^H \mathbf{H}_k^H \mathbf{u}_{k,j}}{\mathbf{u}_{k,j}^H \mathbf{J}_{k,j} \mathbf{u}_{k,j}} \\ &= 1 - \frac{\left(q_{k,j} \mathbf{v}_{k,j}^H \mathbf{H}_k^H \mathbf{J}^{-1} \mathbf{H}_k \mathbf{v}_{k,j} \right)^2}{q_{k,j} \mathbf{v}_{k,j}^H \mathbf{H}_k^H \mathbf{J}^{-1} \mathbf{H}_k \mathbf{v}_{k,j}} \\ &= 1 - q_{k,j} \mathbf{v}_{k,j}^H \mathbf{H}_k^H \mathbf{J}^{-1} \mathbf{H}_k \mathbf{v}_{k,j} = \epsilon_{k,j}^{UL}. \end{aligned} \quad (14)$$

This relationship is similar to one shown for MMSE detection in CDMA systems [39]. By applying (14) to (11), we see that

$$\sum_{k=1}^K \sum_{j=1}^{L_k} \log(1 + \gamma_{k,j}^{UL}) = -\log \left(\prod_{k=1}^K \prod_{j=1}^{L_k} \epsilon_{k,j}^{UL} \right).$$

Since the constraints on $\mathbf{v}_{k,j}$ and $q_{k,j}$ are identical in (11) and (13), the problem of maximizing sum rate in (11) is therefore equivalent to minimizing the PMSE in (13). ■

Note that this result has been independently derived in [32], [33].

C. Algorithm: PMSE Minimization

We now present an algorithm that minimizes the product of mean squared errors. The algorithm draws upon previous work based on uplink-downlink MSE duality [26], [27], which states that all achievable MSEs in the *uplink* for a given \mathbf{U} , \mathbf{V} , and \mathbf{q} (with sum power constraint $\|\mathbf{q}\|_1 \leq P_{\max}$), can also be achieved by a power allocation \mathbf{p} in the *downlink* (where $\|\mathbf{p}\|_1 \leq P_{\max}$). It operates by iteratively obtaining the downlink precoder matrix \mathbf{U} and power allocations \mathbf{p} and the virtual uplink precoder matrix \mathbf{V} and power allocations \mathbf{q} . Each step minimizes the objective function by modifying one of these four variables while leaving the remaining three fixed.

1) *Downlink Precoder*: For a fixed set of virtual uplink precoders \mathbf{V}_k and power allocation \mathbf{q} , the optimum virtual uplink decoder \mathbf{U} is defined by (12). Each ϵ_{kj} is minimized individually by this MMSE receiver, thereby also minimizing the product of MSEs. This \mathbf{U} is normalized and used as the downlink precoder.

2) *Downlink Power Allocation*: The downlink power allocation \mathbf{p} is given by [27]:

$$\mathbf{p} = \sigma^2(\mathbf{D}^{-1} - \mathbf{\Psi})^{-1}\mathbf{1},$$

where $\mathbf{\Psi}$ is the $L \times L$ cross coupling matrix defined as

$$[\mathbf{\Psi}]_{ij} = \begin{cases} |\tilde{\mathbf{h}}_i^H \mathbf{u}_j|^2 = |\mathbf{u}_j^H \tilde{\mathbf{h}}_i|^2 & i \neq j \\ 0 & i = j \end{cases},$$

$$\mathbf{D} = \text{diag} \left\{ \frac{\gamma_{11}^{UL}}{|\mathbf{v}_{11}^H \mathbf{H}_1^H \mathbf{u}_{11}|^2}, \dots, \frac{\gamma_{KLK}^{UL}}{|\mathbf{v}_{KLK}^H \mathbf{H}_K^H \mathbf{u}_{KLK}|^2} \right\},$$

where $\tilde{\mathbf{H}} = \mathbf{H}\mathbf{V} = [\tilde{\mathbf{h}}_1, \dots, \tilde{\mathbf{h}}_L]$, $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_L]$, and $\mathbf{1}$ is the all-ones vector of the required dimension.

3) *Virtual Uplink Precoder*: Given a fixed \mathbf{U} and \mathbf{p} , the optimal decoders \mathbf{V}_k are the MMSE receivers:

$$\mathbf{V}_k = \mathbf{J}_k^{-1} \mathbf{H}_k^H \mathbf{U}_k \sqrt{\mathbf{P}_k}.$$

In this equation, $\mathbf{J}_k = \mathbf{H}_k^H \mathbf{U}_k \mathbf{P}_k \mathbf{U}_k^H \mathbf{H}_k + \sigma^2 \mathbf{I}_{N_k}$ is the receive covariance matrix for user k . The optimum virtual uplink precoders are then the normalized columns of \mathbf{V}_k .

4) *Virtual Uplink Power Allocation*: The power allocation problem on the virtual uplink solves (13) with a fixed matrix \mathbf{V} . While it is well accepted that the power allocation subproblem in PMSE minimization (or equivalently, in sum rate maximization) is non-convex [30], [31], [40], recent work [32] has shown that the optimal power allocation can be found by formulating the subproblem as a Geometric Programming (GP) problem [41]. A similar approach was proposed in [31], where iterations of the the sum rate maximization problem are solved by local approximations of the non-convex sum rate function as a GP. We employ numerical techniques (SQP) to solve the power allocation subproblem.

In summary, the PMSE minimization algorithm keeps three of four parameters (\mathbf{U} , \mathbf{p} , \mathbf{V} , \mathbf{q}) fixed at each step and obtains the optimal value of the fourth. Convergence of the overall algorithm to a local minimum is guaranteed since the PMSE objective function is non-increasing at each of the four parameter update steps. Termination of the algorithm is determined by the selection of a convergence threshold ε .

TABLE I
ITERATIVE PMSE MINIMIZATION ALGORITHM

Iteration:	
1- Downlink Precoder	$\tilde{\mathbf{U}}_k = \mathbf{J}^{-1} \mathbf{H}_k^H \mathbf{V}_k \sqrt{\mathbf{Q}_k}, \quad \mathbf{u}_{kj} = \frac{\tilde{\mathbf{u}}_{kj}}{\ \tilde{\mathbf{u}}_{kj}\ _2}$
2- Downlink Power Allocation via MSE duality	$\mathbf{p} = \sigma^2(\mathbf{D}^{-1} - \mathbf{\Psi})^{-1}\mathbf{1}$
3- Virtual Uplink Precoder	$\tilde{\mathbf{V}}_k = \mathbf{J}_k^{-1} \mathbf{H}_k^H \mathbf{U}_k \sqrt{\mathbf{P}_k}, \quad \mathbf{v}_{kj} = \frac{\tilde{\mathbf{v}}_{kj}}{\ \tilde{\mathbf{v}}_{kj}\ _2}$
4- Virtual Uplink Power Allocation	$\mathbf{q} = \arg \min_{\mathbf{q}} \prod_{k=1}^K \prod_{j=1}^{L_k} \epsilon_{kj}, \text{ s.t. } q_{kj} \geq 0, \ \mathbf{q}\ _1 \leq P_{\max}$
5- Repeat 1-4 until	$[\text{PMSE}_{\text{old}} - \text{PMSE}_{\text{new}}] / \text{PMSE}_{\text{old}} < \varepsilon$

Since the overall minimization problem (13) is not convex, all of the suggested methods are guaranteed to converge only to a local minimum. Nonetheless, simulations suggest that the locally optimal value of the sum rate is not overly sensitive to selection of an appropriate initialization point. It is important to ensure that the initial solution allocates power to all L substreams, as the iterative algorithm tends to not allocate power to streams with zero power. A reasonable initialization is to select random unit-norm precoder vectors in \mathbf{U} and uniform power allocated over all substreams. A summary of our proposed algorithm can be found in Table I.

V. NUMERICAL EXAMPLES

In this section, we present simulation results to illustrate the performance of the proposed algorithms. In all cases, the fading channel is modelled as flat and Rayleigh, with i.i.d. channel coefficients distributed as $\mathcal{CN}(0, 1)$. The examples use a maximum transmit power of $P_{\max} = 1$; SNR is controlled by varying the receiver noise power σ^2 . As stated earlier, the transmitter is assumed to have perfect knowledge of the channel matrix \mathbf{H} .

A. Sum Capacity and Achievable Sum Rate

We first compare the sum rate achievable using linear precoding to the information theoretic capacity of the BC. That is, we consider the spectral efficiency (measured in bps/Hz) that could be achieved under ideal transmission by drawing transmit symbols from a Gaussian codebook. Figure 2 illustrates how the proposed schemes perform when compared to the sum capacity for the broadcast channel (i.e., using dirty paper coding (DPC) [2]) and to linear precoding methods based on channel orthogonalization, i.e., block diagonalization (BD) and zero forcing (ZF) [15].¹ The convergence threshold for the PMSE algorithm is set at $\varepsilon = 10^{-6}$. Note that curves for THP can not be included for comparison, as the modulo and shaping losses from the DPC sum capacity are fundamentally related to THP's nonlinear modulation scheme.

¹Simulation results for the DPC, BD, ZF, and NuSVD plots were obtained by using the `cvx` optimization package [42], [43].

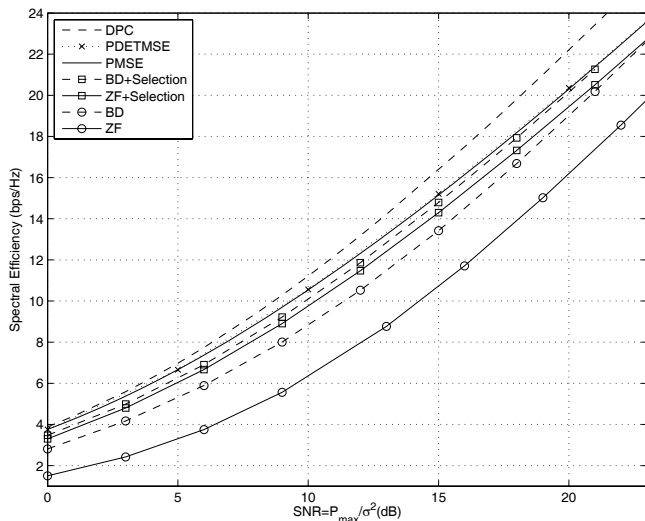


Fig. 2. Comparing PDetMSE, PMSE, DPC and orthogonalization-based methods, $K = 2$, $M = 4$, $N_k = 2$, $L_k = 2$

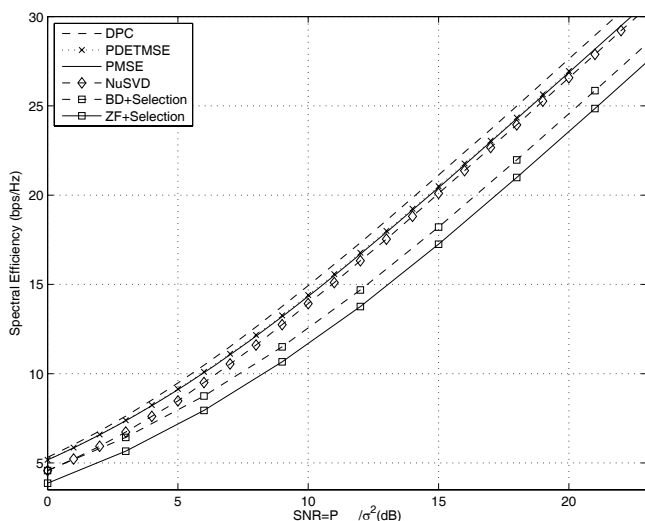


Fig. 3. Comparing PDetMSE, PMSE, DPC and orthogonalization-based methods, $K = 2$, $M = 4$, $N_k = 4$, $L_k = 2$

The simulations in Fig. 2 model a $K = 2$ user system with $M = 4$ transmit antennas and $N_k = 2$ receive antennas per user. We see a negligible difference in performance when comparing the PDetMSE algorithm to the PMSE solution. This is interesting because the relationship between PDetMSE and PMSE mirrors that of BD and ZF; that is, the PDetMSE can be viewed as the block-matrix formulation of the PMSE problem. There is, however, a significant performance difference between BD and ZF. This result is also gratifying because it suggests that the marginal gains achieved by joint processing do not merit the greatly increased computational complexity; the feasible PMSE solution can be used without a large penalty in performance. The PMSE and PDetMSE algorithms do demonstrate a divergence in performance from the theoretical DPC bound at higher SNR. This drop in spectral efficiency may reflect a fundamental gap between the (optimal) nonlinear DPC capacity and the rate achievable under linear precoding, but it may also be caused by the algorithms' convergence to

local minima due to the non-convexity of the optimization problems.

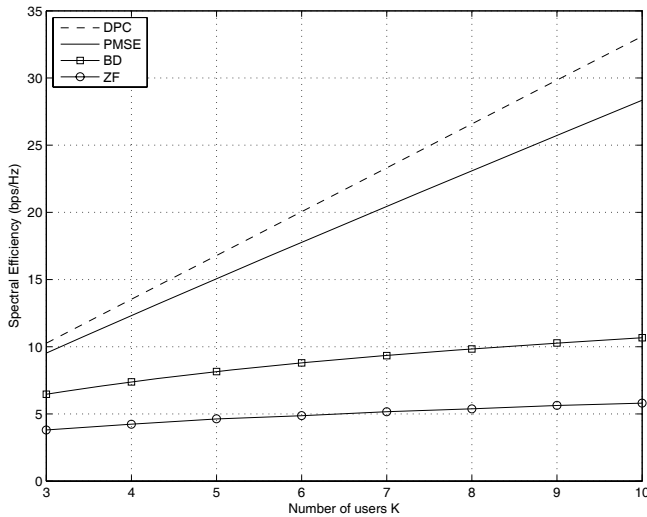
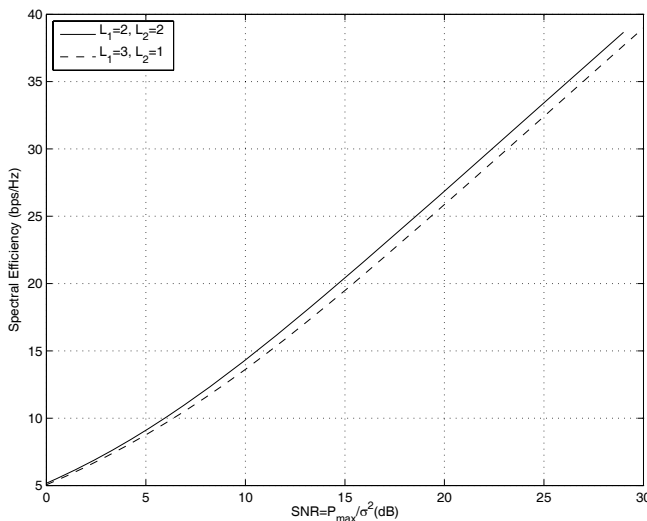
The PMSE algorithm outperforms the BD and ZF methods over the entire SNR range when the orthogonalization-based schemes are forced to use all N receive antennas. However, this approach to orthogonalization is suboptimal; the optimal BD and ZF precoders may be found by selecting the best precoder from all $\sum_{k=1}^{\min(N,M)} \binom{N}{k}$ possible subsets of receive antennas. At high SNR, the PMSE and PDetMSE precoders perform equivalently to the BD precoder with selection; we have observed that the PMSE and PDetMSE precoders (in conjunction with the MMSE receivers) behave like the BD precoder in orthogonalizing the channel at high SNR. The biggest gain in performance over orthogonalization-based solutions occurs at low to mid-SNR values, where BD and ZF suffer due to noise enhancement.

Figure 3 presents simulation results for a similar system as Fig. 2, but with $N_k = 4$ receive antennas per user. In this system, there are fewer transmit antennas than receive antennas ($M < N$), so BD/ZF can not be employed without selection. We include simulation results for BD/ZF with selection, but note the large computational complexity required (selecting the best of 162 candidate precoders). We compare these results to a generalized orthogonalization based approach, referred to as nullspace-directed SVD (NuSVD) in [18], and observe a large difference in performance at high SNR. This gain in spectral efficiency can be attributed to NuSVD's ability to use all $N = 8$ receive antennas, whereas BD and ZF are limited by an antenna constraint.

Once again, Fig. 3 illustrates that the PMSE/PDetMSE approaches outperform orthogonalization, particularly at low to mid-SNR values. This improvement in performance comes at the expense of additional complexity. Even though NuSVD and PMSE/PDetMSE are iterative algorithms, NuSVD requires only one (concave) waterfilling power allocation after convergence of precoder direction iterations, whereas the PMSE/PDetMSE minimization methods employ numerical optimization algorithms (SQP) in each iteration.

Figure 4 shows how the sum throughput scales with the number of users K , for $M = 2K$ transmit antennas and $N_k = 2$ receive antennas per user at 5 dB average SNR. The number of transmit antennas M is chosen so that BD and ZF can be implemented without selection, as selection-based BD and ZF are exponentially complex with $4^K - 1$ possible precoders. This plot illustrates how the proposed scheme takes advantage of the available degrees of freedom at the transmitter and provides throughput significantly better than the orthogonalization based BD and ZF schemes.

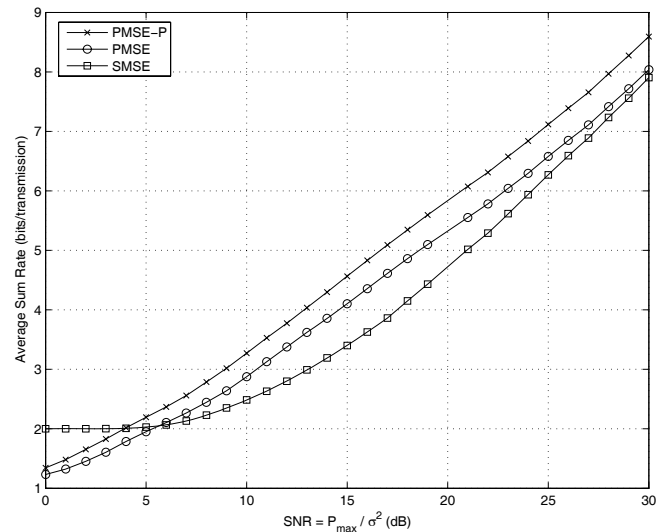
The PMSE and PDetMSE algorithms do not require the explicit selection of L_k ; rather, this parameter is determined implicitly by the power allocation. However, we can force the PMSE algorithm to allocate a maximum number of substreams L_k to each user to gain further insight into its behaviour. In Fig. 5, the number of streams in the $N_k = 4$ system described above is varied from $L_1 = L_2 = 2$ to $L_1 = 3$ and $L_2 = 1$. The achievable sum rate in this system decreases in the latter case, as the asymmetric stream allocation does not correspond to the symmetric (statistically identical) channel configuration. In

Fig. 4. Scaling of sum rate with K , $M = 2K$, $N_k = L_k = 2$, $\text{SNR} = 5\text{dB}$ Fig. 5. Data stream allocation in PMSE optimization, $K = 2$, $M = 4$, $N_k = 4$

this case, user 2 is restricted to only a single data stream, and thus can not take full advantage of good channel realizations. If the goal is always maximizing the sum rate, the users should be allocated the maximum number of data streams in as balanced a manner as possible. Note however that the PMSE algorithm can provide unbalanced allocations if desired for other reasons (e.g., quality of service provisioning).

B. Implementation: Adaptive Modulation

In contrast to the prior results based on Gaussian codebooks, we now consider the selection of constellations for modulation to achieve a maximum throughput for a specified bit error rate (BER) target of $\beta_{k,j}$ on user k 's j^{th} substream. Since the PMSE algorithm assumes unit energy symbols, we use M -PSK constellations in our implementation. Note that the prior assumption of Gaussian noise-plus-interference still holds for a sufficient number of interferers under the central limit theorem. We propose two approaches for selecting the modulation scheme for each substream.

Fig. 6. Sum rate vs. SNR for user 1 with M-PSK modulation, $K = 2$, $M = 4$, $N_k = L_k = 2$

1) *Naive Approach*: This approach selects the largest PSK constellation of $b_{k,j}$ bits per stream that satisfies the required BER constraint. The constraint is satisfied using a closed form BER approximation [44],

$$\text{BER}_{\text{PSK}}(\gamma) \approx c_1 \exp\left(\frac{-c_2\gamma}{2c_3b - c_4}\right), \quad (15)$$

where $M = 2^b$ is the size of the PSK constellation. We apply the least aggressive of the bounds proposed in [44] by using the values $c_1 = 0.25$, $c_2 = 8$, $c_3 = 1.94$, and $c_4 = 0$. We note that this approximation only holds for $b \geq 2$; as such, one can use the exact expression for BPSK:

$$\text{BER}_{\text{BPSK}}(\gamma) = \frac{1}{2} \text{erfc}(\sqrt{\gamma}). \quad (16)$$

The BPSK expression can be used as a test of feasibility for the specified BER target; if the resulting BER under BPSK modulation is higher than $\beta_{k,j}$, then we have two options: either declare the BER target infeasible, or transmit using the lowest modulation depth available (i.e. BPSK). In this work, we have elected to transmit using BPSK whenever the PMSE stage has allocated power to the data stream.

2) *Probabilistic Approach*: The naive approach is quite conservative in that there may be a large gap between the BER requirement and $\text{BER}_{b_{k,j}}$, the BER achieved for each channel realization. We suggest a *probabilistic* bit allocation scheme that switches between $b_{k,j}$ bits (as determined by the naive approach) and $b_{k,j} + 1$ bits with probability $p_{k,j} = [\beta_{k,j} - \text{BER}_{b_{k,j}}] / [\text{BER}_{b_{k,j}+1} - \text{BER}_{b_{k,j}}]$. This modulation strategy may not be appropriate for systems requiring instantaneous satisfaction of BER constraints; however, the probabilistic method will still achieve the desired BER in the long-term over multiple channel realizations.

Figure 6 shows the sum rate achieved in the same system configuration as in Fig. 2 ($K = 2$, $M = 4$, $N_k = 2$) under the M -PSK modulation scheme described above. The simulations use two data streams per user and a target bit error rate of $\beta_{k,j} = 10^{-2}$; 5000 data and noise realizations are used

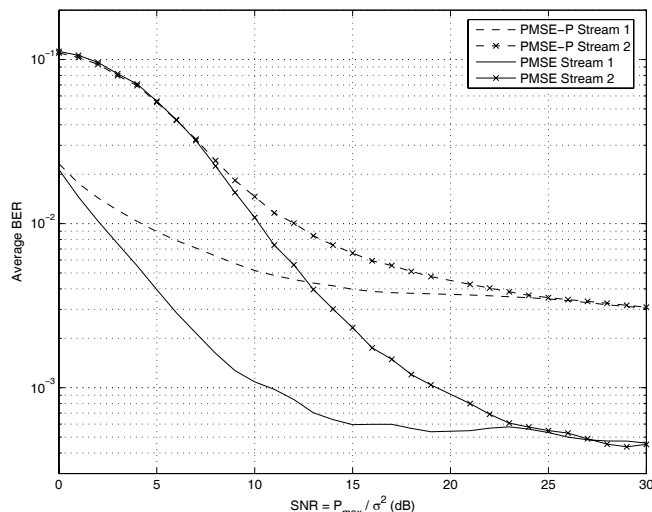


Fig. 7. BER vs. SNR for user 1 with M-PSK modulation, $K = 2$, $M = 4$, $N_k = L_k = 2$

for each channel realization. The plot illustrates the average number of bits per transmission for user 1; due to symmetry, the corresponding plot for user 2 is identical. Note that in contrast to the previous results based on Gaussian coding using spectral efficiency, the sum rate in Fig. 6 is the average number of bits transmitted per realization using symbols from a PSK constellation.

In Fig. 6, we consider using the PSK modulation scheme for the PMSE precoder and the SMSE precoder designed in [27]. Examination of this plot reveals that using the PMSE criterion is justified at practical SNR values with improvements of approximately one bit per transmission near 15 dB. Furthermore, using the probabilistic modulation scheme (designated “PMSE-P”) yields an additional improvement of more than half a bit per transmission across all SNR values.

In Fig. 7, we plot average BER versus SNR for the same system configuration as in Fig. 6. This plot illustrates how the naive bit allocation algorithm attempts to achieve the target BER of 10^{-2} for all data streams under PMSE, but also overshoots the target, converging to a BER of approximately 5×10^{-4} . This can be attributed to the looseness of the BER bound mentioned above. In contrast, the probabilistic rate allocation algorithm not only increases the rate, as shown in Fig. 6, but also converges to a BER that is much closer to the desired target BER. The remaining gap between the actual BER achieved and the target BER can again be attributed to looseness in the approximations of (15) and (16).

VI. CONCLUSIONS

In this paper, we have considered the problem of designing a linear precoder to maximize sum throughput in the multiuser MIMO downlink under a sum power constraint. We have compared the maximum achievable sum rate performance of linear precoding schemes to the sum capacity in the general MIMO downlink, without imposing constraints on the number of users, base station antennas, or mobile antennas. The problem was reformulated in terms of MSE based expressions, and the joint processing solution based on PDetMSE minimization

was shown to be theoretically optimal, but computationally infeasible. A suboptimal framework based on scalar (per-stream) processing was then proposed, and an implementation was provided based on PMSE minimization and employing a known uplink-downlink duality of MSEs. We evaluated the performance of these schemes in the context of orthogonalizing approaches, which suffer from noise enhancement, and have shown that the MSE based optimization schemes are able to achieve significant performance improvements. Furthermore, we have demonstrated that negligible performance losses occur when using the suboptimal PMSE criterion in comparison to the optimum PDetMSE criterion.

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