THE EFFECT OF NOISE IN THE DATA ON THE CAUCHY METHOD*

R. Adve and T. Sarkar
Department of Electrical and Computer Engineering
Syracuse University
Syracuse, New York 13244

KEY TERMS

Data noise, numerical simulation, probability density function

ABSTRACT

In this article the effect of noise in the data on the Cauchy method is discussed. The noise is assumed to be zero mean and Gaussian. The resulting random variable is a ratio of two Gaussians. The theoretical probability density function is verified by numerical simulation.

1. INTRODUCTION

The Cauchy method [1] has been shown to provide accurate broadband information from narrowband data. The method deals with approximating a function by a ratio of two polynomials. Given the value of the function and its derivatives at a few points, the order of the polynomials and their coefficients are evaluated. Once the coefficients of the two polynomials are known, they can be used to generate the parameter over the entire band of interest. The Cauchy method was shown to be applicable to the cases where the input data were measured values of the function and not theoretical values.

However, no measuring instrument is perfect. Each measurement has, added to the signal, an unwanted noise component. Reference [1] does not discuss how this noise affects the results from the Cauchy method. The presence of noise in the data limits the effectiveness of the method. In this article we try to quantify the limitations of the Cauchy method when the input data are subject to contamination by noise.

Throughout this article we assume that the noise is additive, stationary, zero mean, and Gaussian. This assumption is approximately valid for most measuring instruments. Using this assumption, the probability density function (PDF) of the parameter, as a function of frequency, is evaluated. This is compared to the PDF approximated by a computer numerical simulation.

To make the problem tractable, certain simplifying assumptions are necessary. This includes assuming that the coefficients of the polynomials in the Cauchy method are independent random variables. As the theory will show, this assumption is not strictly true. However, the error introduced in the PDF due to this assumption is minimal.

Another assumption made is that the noise affects each measurement independently. The noise is also assumed to affect each measurement, on average, equally. This means, that the average power in the noise in each measurement is assumed constant over repeated measurements. This assumption too is approximately valid for most measuring systems.

Using these assumptions we derive the theoretical PDF of the estimate of the parameter as a function of frequency. The theoretical PDF was verified by numerical simulations.

2. REVIEW OF THE CAUCHY METHOD

The Cauchy method approximates a system function $H(s)$ with a ratio of two polynomials. Hence, consider

$$H(s) \simeq \frac{A(s)}{B(s)} = \frac{\Sigma_{k=0}^{P} a_k s^k}{\Sigma_{k=0}^{Q} b_k s^k}. \tag{1}$$

Here, the given information is assumed to be the N measured values of the function (H) at frequency points s_j, $j = 1, \ldots, N$. In this case, the Cauchy problem is

Given $H(s_j)$ for $j = 1, \ldots, N$, find P, Q, $\{a_k, k = 0, \ldots, P\}$, and $\{b_k, k = 0, \ldots, Q\}$.

The approach is to enforce the equality of Eq. (1) at the points of measurement s_j. Hence, one obtains

$$A(s_j) = H(s_j)B(s_j), A(s_j) - H(s_j)B(s_j) = 0. \tag{2}$$

Using the polynomial expansions for $A(s)$ and $B(s)$, and the notation $H_j = H(s_j)$, Eq. (2) yields

$$a_0 + a_1 s_j + a_2 s_j^2 \cdots a_P s_j^P - H_j b_0 - H_j b_1 s_j \cdots - H_j b_Q s_j^Q = 0, \tag{3}$$

for $j = 1, \ldots, N$.

Writing this equation in matrix form, we get

$$[\mathbf{C}] \begin{bmatrix} a \\ b \end{bmatrix} = 0, \tag{4}$$

where

$$[\mathbf{C}] = \begin{bmatrix} 1 & s_1 \ldots s_1^P & -H_1 & -H_1 s_1 \ldots - H_1 s_1^Q \\ 1 & s_2 \ldots s_2^P & -H_2 & -H_2 s_2 \ldots - H_2 s_2^Q \\ \vdots & \vdots & \vdots & \vdots \\ 1 & s_N \ldots s_N^P & -H_N & -H_N s_N \ldots - H_N s_N^Q \end{bmatrix} \tag{5}$$

$$[a] = [a_0, a_1, a_2 \cdots a_P]^T \tag{6}$$

and

$$[b] = [b_0, b_1, b_2 \cdots b_Q]^T. \tag{7}$$

The matrix $\mathbf{C}$ is of order $N \times P + Q + 2$. The solution for $\{a_k\}$ and $\{b_k\}$ is unique if the total number of samples is greater than or equal to the total number of unknown coefficients $P + Q + 2$ [1]; that is,

$$N \equiv \sum_{j=1}^{J} (N_j + 1) \geq P + Q + 2.$$

---

*This work was supported in part by Scientific Atlanta and in part by the CASE Center at Syracuse University.

A singular value decomposition (SVD) of the matrix **C** will give us a gauge of the required values of P and Q [2]. A SVD results in the equation

$$[\mathbf{U}][\mathbf{\Sigma}][\mathbf{V}]^T \begin{bmatrix} a \\ b \end{bmatrix} = 0. \tag{8}$$

The matrices **U** and **V** are unitary matrices and **Σ** is a diagonal matrix with the singular values of **C** in descending order as its entries. The columns of **U** are the left eigenvectors of **C** or the eigenvectors of CC^T. The columns of **V** are the right eigenvectors of **C** or the eigenvectors of C^TC. The singular values are the square roots of the eigenvalues of the matrix C^TC. Therefore, the singular values of any matrix are real and positive. The number of nonzero singular values is the rank of the matrix in Eq. (8) and so gives us an idea of the information in this system of simultaneous equations. If R is the number of nonzero singular values, the dimension of the right null space of **C** is $P + Q + 2 - R$. Our solution vector belongs to this null space. Hence, to make this solution unique, we need to make the dimension of this null space 1 so that only one vector defines this space. Hence P and Q must satisfy the relation

$$R + 1 = P + Q + 2. \tag{9}$$

Hence, the solution algorithm must include a method to estimate R. This is done by starting out with the choices of P and Q that are higher than can be expected for the system at hand. Then we get an estimate for R from the number of nonzero singular values of the matrix **C**. Now, using Eq. (9) we get better estimates for P and Q. Letting P and Q stand for these new estimates of the polynomial orders, we can recalculate the matrix **C**. Therefore,

$$[\mathbf{C}] \begin{bmatrix} a \\ b \end{bmatrix} = 0. \tag{10}$$

[**C**] is a rectangular matrix with more rows than columns. Another SVD of this matrix brings us back to the equation

$$[\mathbf{U}][\mathbf{\Sigma}][\mathbf{V}]^T \begin{bmatrix} a \\ b \end{bmatrix} = 0 \tag{11}$$

This homogeneous matrix equation can be solved by the total least squares (TLS) [3]. By the theory of the TLS, the solution is proportional to the last column of the matrix **V**. Since any constants of proportionality cancel out while dividing the two polynomials, we can choose

$$\begin{bmatrix} a \\ b \end{bmatrix} = [\mathbf{V}]_{P+Q+2}. \tag{12}$$

Using this solution for the coefficients, the desired parameter can be approximated at any frequency point of interest.

3. THE EFFECT OF NOISE ON THE SOLUTION VECTOR

As seen above, the solution vector belongs to the invariant subspace that is spanned by the right singular vector $[\mathbf{V}]_{P+Q+2}$. This singular vector is associated with the smallest singular value. However, because of the noise in the data, the entries of matrix **C** are perturbed from their true values. Hence, the solution vector is also perturbed. We need to quantify the perturbation of this subspace.

Notation. In this article a perturbed parameter or matrix will be represented by a tilde (~) above the corresponding unperturbed parameter or matrix.

3.1. Perturbation of Invariant Subspaces. Let $\Re$ denote the set of real numbers, $\Re^n$ the set of real vectors of length n, and $\Re^{n\times p}$ the set of real matrices of order $n \times p$.

Consider an arbitrary matrix $A \in \Re^{N\times P}$ with $P \leq N$. Let $\bar{A} = A + E$, where E is the perturbation to the matrix A, and

$$[U^T][A][V] = \begin{array}{cc} \left(\begin{array}{cc} \sigma_1 & 0 \\ 0 & \Sigma_2 \\ 0 & 0 \end{array}\right) & \begin{array}{c} 1 \\ P-1 \\ N-P \end{array} \\ \begin{array}{cc} 1 & P-1 \end{array} & \end{array} \tag{13}$$

Here the figures below the matrix indicate the number of columns in each submatrix, while the figures to the side of the matrix indicate the number columns in each submatrix. Also,

$$\mathbf{U} = (u_1 \mid U_2 \mid U_3),$$

$$\mathbf{V} = (v_1 \mid V_2).$$

Here, $u_1 \in \Re^N$, $U_2 \in \Re^{N\times(P-1)}$, $U_3 \in \Re^{N\times(N-P)}$, $v_1 \in \Re^P$, and $V_2 \in \Re^{P\times(P-1)}$. σ_1 is the singular value corresponding to the left singular vector u_1 and the right singular vector v_1. This singular value can be the one of interest and not just the largest singular value. In the Cauchy method the singular value of interest is the smallest or the zero singular value. Σ_2 is the diagonal matrix with the rest of the singular values of C as its entries. These singular values can be ordered arbitrarily as long as the columns of U and V are permuted appropriately so as to maintain the equality of Eq. (13).

If

$$[U]^T[E][V] = \begin{pmatrix} \gamma_{11} & g_{12}^T \\ g_{21} & G_{22} \\ g_{31} & G_{32} \end{pmatrix}, \tag{14}$$

where $\gamma_{11} \in \Re$, $g_{12}, g_{21}, g_{31} \in \Re^{P-1}$, $G_{22} \in \Re^{P-1\times P-1}$, and $G_{32} \in \Re^{N-P-1\times P-1}$, and if σ_1 is not repeated as a singular value, then [4]

$$\bar{v}_1 = v_1 + V_2(\sigma_1^2 I - \Sigma_2^2)^{-1} h + O(\|E\|_2^2), \tag{15}$$

where

$$h = \sigma_1 g_{12} + \Sigma_2 g_{21}.$$

3.2. Perturbation of the Solution in the Cauchy Method. In all measurements the true value of the measured parameter (here H_i) is perturbed by an additive noise component. Hence,

$$\tilde{H}_i = H_i + e_i,$$

where $\tilde{H}_i$ is the value of H_i after it has been perturbed by noise e_i.

In the following discussion we are assuming

1. The noise is only in the measurement of the parameter $[H(s)]$, not in the measurement of the frequency (s).
2. $\mathbf{C}x = 0$ has a solution which is unique to within a constant.
   (a) $x = v_1$, with $\sigma_1 = 0$
   (b) $\sigma_1 = 0$ is a simple singular value. This assumption is valid because in the solution procedure we made sure that the rank of the null space of $\mathbf{C}$ is one.
3. $\tilde{H}(s_i) = H(s_i) + e_i$, $\{e_i\}_{i=1}^N$ are zero mean, Gaussian, uncorrelated, and have equal variances σ^2.

Using the foregoing notation for a perturbed matrix and Eq. (10), we get

$$[\tilde{\mathbf{C}}] \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} \simeq 0, \tag{16}$$

where

$$[\tilde{\mathbf{C}}] = \begin{bmatrix} 1 & s_1 \cdots s_1^P & -\tilde{H}_1 & -\tilde{H}_1 s_1 \cdots -\tilde{H}_1 s_1^Q \\ 1 & s_2 \cdots s_2^P & -\tilde{H}_2 & -\tilde{H}_2 s_2 \cdots -\tilde{H}_2 s_2^Q \\ \vdots & \vdots & \vdots & \vdots \\ 1 & s_N \cdots s_N^P & -\tilde{H}_N & -\tilde{H}_N s_N \cdots -\tilde{H}_N s_N^Q \end{bmatrix}, \tag{17}$$

where
P = estimate of the order of the numerator,
Q = estimate of the order of the denominator,
N = number of sample points.

$$\Rightarrow [\tilde{\mathbf{C}}] = [\mathbf{C}] + [\mathbf{E}], \tag{18}$$

where $\mathbf{E}$ is the additive error to the matrix $\mathbf{C}$ due to noise in the data. Hence,

$$[E] = [\mathbf{0} \mid \mathbf{E}_1], \tag{19}$$

where $[\mathbf{0}]$ is a zero matrix of order $N \times P + 1$ and

$$[E_1] = -\begin{bmatrix} e_1 & e_1 s_1 & e_1 s_1^2 \cdots e_1 s_1^Q \\ e_2 & e_2 s_2 & e_2 s_2^2 \cdots e_2 s_2^Q \\ \vdots & \vdots & \vdots \\ e_N & e_N s_N & e_N s_N^2 \cdots e_N s_N^Q \end{bmatrix}_{N \times Q+1} \tag{20}$$

$$E_1 = -\underbrace{\begin{bmatrix} e_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_N \end{bmatrix}}_{N \times N} \underbrace{\begin{bmatrix} 1 & s_1 & s_1^2 & \cdots & s_1^Q \\ 1 & s_2 & s_2^2 & \cdots & s_2^Q \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & s_N & s_N^2 & \cdots & s_N^Q \end{bmatrix}}_{N \times Q + 1} \tag{21}$$

$$[U]^T[E][V] = [U]^T[0|E_1] \begin{array}{cc} \left[\begin{array}{c|c} v_1' & V_2' \\ v_1'' & V_2'' \end{array}\right] & \begin{array}{c} P+1 \\ Q+1 \end{array} \\ \begin{array}{cc} 1 & \quad P+Q+2 \end{array} & \end{array} \tag{22}$$

$$= [U^T E_1 v_1'' \mid U^T E_1 V_2''] . \tag{23}$$

Using the notation of Section 3.1,

$$[U]^T[E][V] = [U^T E_1 v_1'' \mid U^T E_1 V_2''] = \begin{pmatrix} \gamma_{11} & g_{12}^T \\ g_{21} & G_{22} \\ g_{31} & G_{32} \end{pmatrix} . \tag{24}$$

Because v_1 is the solution of the unperturbed Cauchy equation

$$[\mathbf{C}] \begin{bmatrix} a \\ b \end{bmatrix} = 0,$$

and v_1'' is the vector of the last $Q + 1$ entries of v_1, v_1'' is the vector of denominator coefficients. Also, the singular value of interest (σ_1) is zero. Hence, in the notation of Section 3.1,

$$h = \sigma_1 g_{12} + \Sigma_2 g_{21} \tag{25}$$

$$= \Sigma_2 g_{21} . \tag{26}$$

Also, using Eq. (15)

$$\tilde{v}_1 = v_1 + V_2 \Sigma_2^{-1} g_{21} + O(\|E\|_2^2). \tag{27}$$

Hence, g_{12} is of no consequence.

From Eq. (24),

$$[U]^T[E_1][v_1''] = \begin{pmatrix} \gamma_{11} \\ g_{21} \\ g_{31} \end{pmatrix} . \tag{28}$$

Using Eq. (21), and the fact that v_1'' is the vector of denominator coefficients,

$$E_1 v_1'' = -\begin{bmatrix} e_1 & 0 & 0 & \cdots & 0 \\ 0 & e_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_N \end{bmatrix} \begin{bmatrix} de(s_1) \\ de(s_2) \\ \vdots \\ de(s_N) \end{bmatrix}, \tag{29}$$

where $de(s_i) = \Sigma_{k=0}^{Q} b_k s_i^k$ is the value of the unperturbed denominator polynomial evaluated at s_i. For convenience we define a new vector $\tilde{e}$ as

$$\tilde{e} \equiv E_1 v_1'' = -\begin{bmatrix} e_1\, de(s_1) \\ e_2\, de(s_2) \\ \vdots \\ e_N\, de(s_N) \end{bmatrix} . \tag{30}$$

Using this equation, the fact that $U = [u_1 \mid U_2 \mid U_3]$, and Eq. (28),

$$g_{21} = U_2^T \tilde{e}. \tag{31}$$

Therefore, using Eq. (27),

$$\tilde{v}_1 = v_1 + V_2 \Sigma_2^{-1} U_2^T \tilde{e} + O(\|E\|_2^2). \tag{32}$$

Because the elements of $\tilde{e}$ are Gaussian random variables, $\tilde{v}_1$ is, to the *first order of approximation,* a Gaussian random vector.

Now, using Eq. (13) and the fact that $\sigma_1 = 0$,

$$C = U_2 \Sigma_2 V_2^T$$

$$\Rightarrow V_2 \Sigma_2^{-1} U_2^T = C^+, \tag{33}$$

where C^+ is the pseudoinverse of C. Therefore, to the first order of approximation,

$$\tilde{v}_1 = v_1 + C^+ \tilde{e}. \tag{34}$$

Using the fact that C^+ is unperturbed and the noise is zero mean, the expectation value of the solution vector (v_1) is given by

$$\mathbf{E}(v_1) = v_1 + C^+\mathbf{E}(\tilde{e}) \tag{35}$$

$$= v_1. \tag{36}$$

Here $\mathbf{E}$ is the expectation operator and not the error matrix. Therefore, to the *first order of approximation*, the estimator is *unbiased*.

The covariance matrix of v_1 is given by

$$\mathrm{cov}(v_1) = \mathbf{E}[\tilde{v}_1 - v_1)(\tilde{v}_1 - v_1)^T].$$

Using Eq. (34), we have

$$\mathbf{E}[\tilde{v}_1 - v_1)(\tilde{v}_1 - v_1)^T] = \mathbf{E}[C^+\tilde{e}\tilde{e}^T C^{+T}].$$

$$= C^+\mathbf{E}[\tilde{e}\tilde{e}^T]C^{+T}. \tag{37}$$

Now,

$$\tilde{e}\tilde{e}^T = \begin{bmatrix} e_1\, de(s_1) \\ e_2\, de(s_2) \\ \vdots \\ e_N\, de(s_N) \end{bmatrix} [e_1\, de(s_1) e_2\, de(s_2) \cdots e_N\, de(s_N)]. \tag{38}$$

Therefore, the ijth entry of this matrix is given by

$$[\tilde{e}\tilde{e}^T]_{ij} = e_i e_j\, de(s_i)\, de(s_j). \tag{39}$$

Because e_i and e_j are assumed to be zero mean, independent, and identically distributed with variance σ^2,

$$\mathbf{E}[e_i e_j] = \sigma^2\delta_{ij}, \tag{40}$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbf{E}[\tilde{e}\tilde{e}^T] = \sigma^2 \begin{bmatrix} de^2(s_1) & 0 & 0 & \cdots & 0 \\ 0 & de^2(s_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & de^2(s_N) \end{bmatrix} \tag{41}$$

$$\mathbf{E}[(\tilde{v}_1 - v_1)(\tilde{v}_1 - v_1)^T]$$

$$= \sigma^2 C^+ \begin{bmatrix} de^2(s_1) & 0 & 0 & \cdots & 0 \\ 0 & de^2(s_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & de^2(s_N) \end{bmatrix} C^{+T}. \tag{42}$$

Letting $C^+_{ij} = c_{ij}$, the autocovariance of the ith entry of $\tilde{v}_1$ is given by

$$\mathbf{E}[C^+\tilde{e}\tilde{e}^T C^+]_{ii} = \sigma^2 \sum_{j=1}^{N} c_{ij}^2\, de^2(s_j). \tag{43}$$

This is the variance of the ith entry in the vector of coefficients. Hence, if $i \leq P + 1$, we are dealing with a numerator coefficient, else we are dealing with a denominator coefficient.

Because we have solved a matrix equation in which the elements of the matrix are Gaussian random variables, each element of the solution vector is a Gaussian random variable. Also, the numerator and denominators are linear combinations of the coefficients. Hence, the numerator and denominator are Gaussian random variables *as functions of frequency*. Hence, to completely characterize the numerator and denominator random variables, we only need their expectation values and variances.

To make this problem of the ratio of two Gaussians solvable, we have to assume that any two coefficients are independent of each other. Hence, the cross-covariance matrix of v_1 is assumed to be diagonal.

Now,

$$\tilde{A}(s) = \sum_{k=0}^{P} \tilde{a}_k s^k \tag{44}$$

$$\tilde{B}(s) = \sum_{k=0}^{Q} \tilde{b}_k s^k. \tag{45}$$

Therefore,

$$\mathbf{E}[\tilde{A}(s)] = \sum_{k=0}^{P} \mathbf{E}[\tilde{a}_k]s^k, \tag{46}$$

and

$$\mathbf{E}[\tilde{B}(s)] = \sum_{k=0}^{Q} \mathbf{E}[\tilde{b}_k]s^k. \tag{47}$$

However, because to the first order of approximation the coefficients are unbiased,

$$\mathbf{E}[\tilde{A}(s)] = \sum_{k=0}^{P} a_k s^k \tag{48}$$

and

$$\mathbf{E}[\tilde{B}(s)] = \sum_{k=0}^{Q} b_k s^k. \tag{49}$$

Therefore, the estimators for the numerator and denominator as a function of frequency are unbiased. However, as we will see, because the ratio of two variables is not a linear function, this does not mean the final estimator is unbiased.

To calculate the variances of the numerator and denominator as a function of frequency,

$$\mathrm{var}[A(s)] = \mathrm{var}\left[\sum_{k=0}^{P} \tilde{a}_i s^i\right]. \tag{50}$$

Using the assumption that each coefficient is independent of the others,

$$\mathrm{var}[A(s)] = \sum_{k=0}^{P} \mathrm{var}(\tilde{a}_i)s^{2i}. \tag{51}$$

Therefore, from Eq. (43),

$$\mathrm{var}[A(s)] = \sigma^2 \sum_{i=1}^{P+1} s^{2i} \sum_{j=1}^{N} c_{ij}^2 \, de^2(s_j). \qquad (52)$$

Similarly,

$$\mathrm{var}[B(s)] = \sigma^2 \sum_{i=P+2}^{P+Q+2} s^{2i} \sum_{j=1}^{N} c_{ij}^2 \, de^2(s_j). \qquad (53)$$

Let $\overline{N} \equiv \mathbf{E}[A(s)]$, $\overline{D} \equiv \mathbf{E}[B(s)]$, $a^2 \equiv \mathrm{var}[A(s)]$, and $b^2 \equiv \mathrm{var}[B(s)]$. Therefore, the problem has reduced to: Given the means and variances of two independent Gaussian random variables, what is the PDF of their ratio? This problem has been solved in Reference [5].

In the notation of [5], if N and D are independent Gaussian random variables with means $\overline{N}$ and $\overline{D}$, respectively, and variances a^2 and b^2, respectively, and if

$$R = \frac{N}{D},$$

then the probability density function of R is given by

$$f_R(r) = \sqrt{\frac{(ab)^3}{\pi}} \frac{1}{b^2r^2 + a^2} e^{(N^2/2a^2 + D^2/2b^2)} \times \left[Z\, erf(Z) \exp(Z^2) + \frac{1}{\sqrt{\pi}} \right], \qquad (54)$$

where

$$Z = \frac{1}{\sqrt{2b^2a^2}} \left(\frac{b^2\overline{N}r + a^2\overline{D}}{\sqrt{b^2r^2 + a^2}} \right),$$

and the error function is defined as

$$erf(Z) = \frac{2}{\sqrt{\pi}} \int_0^Z e^{-t^2} \, dt.$$

Hence, we have the theoretical PDF of the ratio of two random variables. However, this density function is an approximation of the true density function. To obtain the true density function we would need to take into account the cross correlation between the coefficients. This leads to a problem that is highly difficult to solve.

4. NUMERICAL EXAMPLES

To test the above theory, the Cauchy method was tested with a simple example. As an example the function chosen to be the testing function was

$$H(s) = \frac{\Sigma_{k=0}^{4} ks^k}{\Sigma_{k=0}^{5} (k + 1)s^k}. \qquad (55)$$

This ratio of two polynomials was evaluated at 31 points in the range $s = 2.0$ and $s = 4.0$. Two tests were performed on these data.

In the first test, Gaussian noise was added to the data directly. A numerical Gaussian random number generator was used. The power in the noise was chosen such that the signal-to-noise ratio (SNR) was 30 dB. These perturbed data were used as inputs to the Cauchy program. The resulting polynomials were used to evaluate the parameter at $s = 3.0$. This was considered to be one sample of the random variable at $s = 3.0$. Samples taken numbered 1001. A PDF estimator was used to estimate the PDF at $s = 3.0$. Figure 1 shows the PDF found using this method. This is the plot marked "Adding noise to the data."

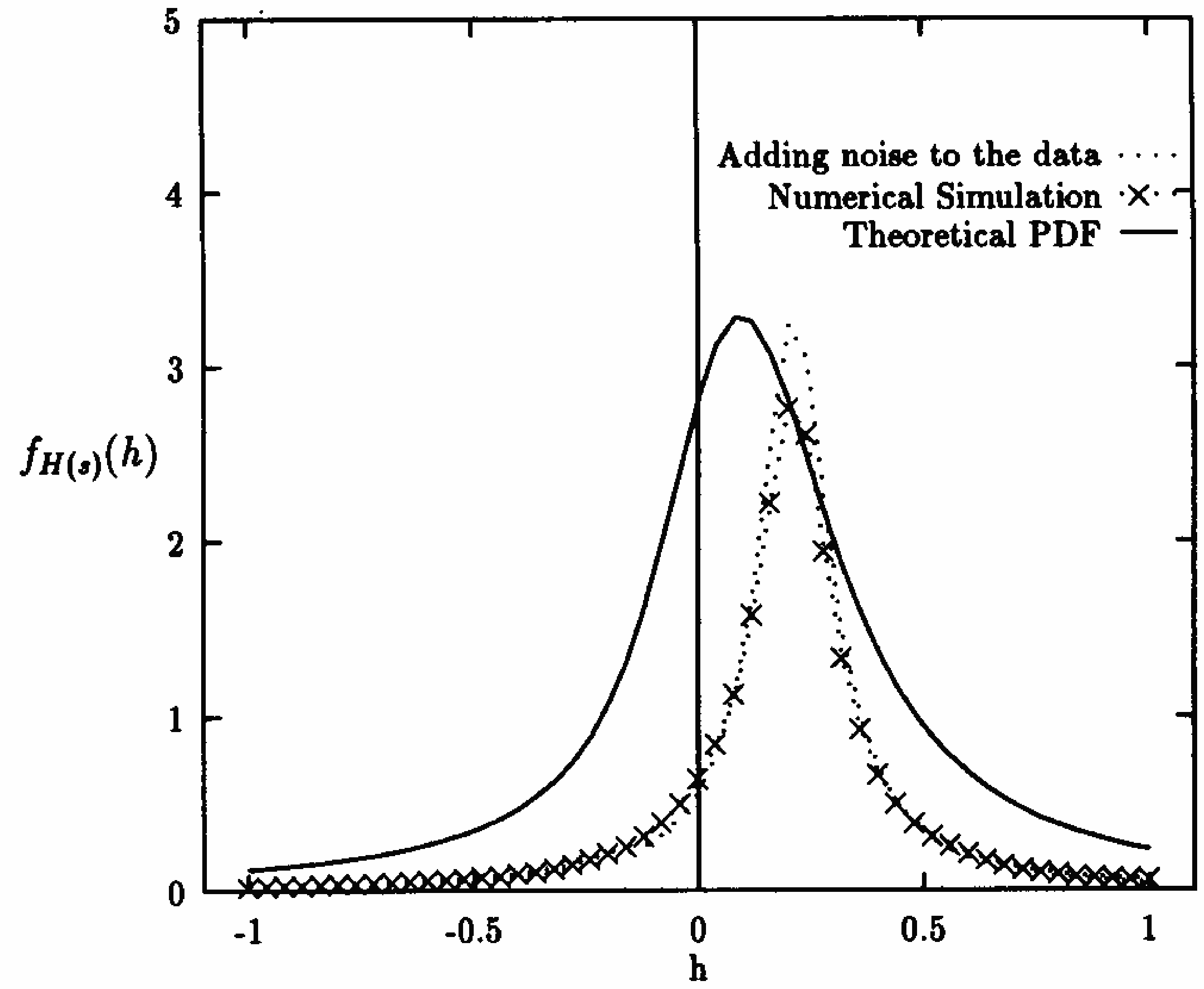

Figure 1 Comparison of theoretical PDF and numerically simulated PDFs. SNR = 30 dB

In the second test, the original unperturbed data between $s = 2.0$ and $s = 4.0$ were used as inputs to the Cauchy program. The unperturbed numerator and denominator coefficients were evaluated. The means of the numerator and denominator were evaluated using Eqs. (48) and (49), respectively. Also, the variances of the numerator and denominator were evaluated using Eqs. (52) and (53), respectively. Using these values of means and variances, a Gaussian random variable, with the numerator mean and variance, was divided with another Gaussian random variable with the denominator mean and variance. This was repeated 1001 times.

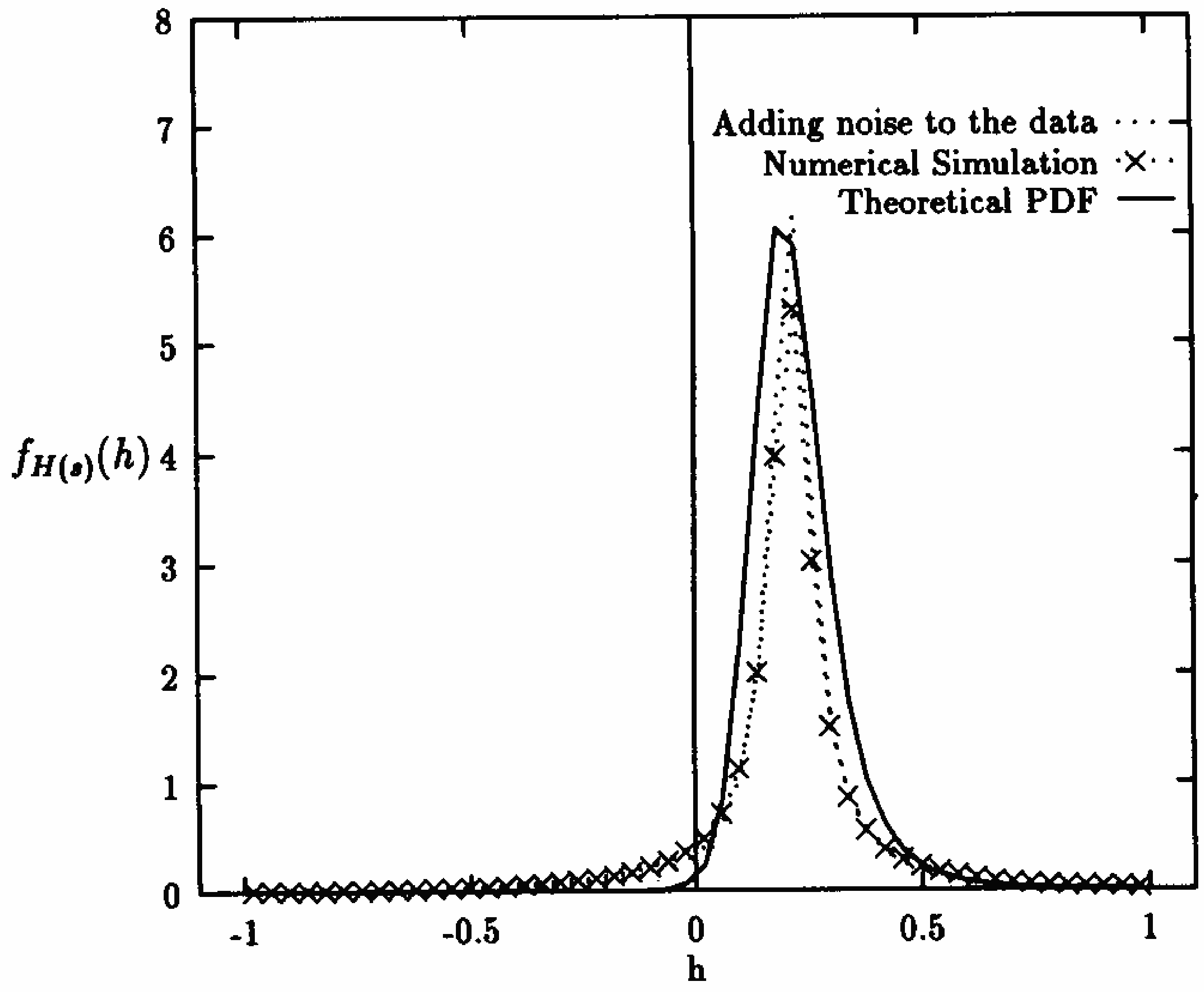

Figure 2 Comparison of theoretical PDF and numerically simulated PDFs. SNR = 40 dB

The Gaussian random numbers were generated using the same random number generator as in the first test.

The 1001 samples gotten from this test were used as input to the same PDF estimator. The result from this estimation of the PDF is shown in Figure 1. This plot is labeled "Numerical Simulation."

Finally, these two PDFs are compared with the theoretical PDF in Eq. (54). The choices of $\overline{N}$, $\overline{D}$, a^2, and b^2 are gotten from the theoretical means and variances used in the second test.

At $s = 3.0$, using the above function

Actual Value: 0.2126
Mean (adding noise to the data): 0.2124
Mean (dividing two Gaussians with the theoretical means and variances): 0.1804

Figure 2 shows the same three PDFs for a signal-to-noise ratio of 40 dB. Here the agreement is better than in the earlier case. This is to be expected, because the assumptions come closer to being satisfied as the noise reduces.

REFERENCES

1. R. S. Adve and T. K. Sarkar, "Generation of Accurate Broadband Information from Narrowband Data Using the Cauchy Method," *Microwave Opt. Technol. Lett.*, Vol. 6, No. 10, Aug. 1993, pp. 569–573.
2. G. H. Golub and C. F. Van Loan, *Matrix Computations* (2nd ed.), Johns Hopkins University Press, Baltimore, 1989.
3. S. Van Huffel, "Analysis of the Total Least Squares Problem and Its Use in Parameter Estimations," Ph.D. dissertation, Department Electrotechnick, Kattrolicke Universiterit Leuven, August 1987.
4. G. W. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic, Orlando, FL, 1990.
5. I. Kanter, "The Ratios of Functions of Random Variables," *IEEE Trans. Aerospace Electron. Syst.*, Vol. AES-13, Nov. 1977, pp. 624–630.

Received 10-18-93

Microwave and Optical Technology Letters, 7/5, 242–247

CCC 0895-2477/94