Fractional Cooperation and the Max-Min Rate in a Multi-Source Cooperative Network

Ehsan Karamad and Raviraj Adve
The Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto
10 King's College Road, Toronto, ON M5S 3G4, Canada
Email: {ekaramad, rsadve}@comm.utoronto.ca

***Abstract*—We maximize the minimum rate of each information flow in a multi-source, multi-relay, single destination cooperative network. The relays use the the decode-and-forward protocol while all transmissions use OFDM. The key to our approach is fractional cooperation wherein not all source subcarriers are relayed. There may, therefore, be fewer relays than sources. Finding the optimal allocation relay subcarriers to sources is combinatorial and has exponential complexity. We develop an upper bound on the max-min rate and present an algorithm which performs very close to this upper bound. Our simulation results show that, accounting for the overhead due to relaying, there is an optimal number of relays which is, in general, less than the number of sources.**

I. Introduction

Cooperative communications incorporates relaying and cooperation between nodes in order to increase the performance, coverage, and spectral efficiency of data networks [1]–[3]. Cooperative communication provides spatial diversity even if each individual node in the network does not possess multiple antennas. Amongst the multiple schemes available, in this paper, we will focus on relaying using the decode-and-forward (DF) scheme [2]. The relay decodes, and then rencodes, the source data. An independent choice is to focus on relaying based on *selection* [4]–[6]. Selection, wherein each source partners with a single 'best' relay' has been shown to provide almost all the benefits of cooperation with minimum overhead and without issues of synchronization.

As data rates rise and multipath fading becomes increasingly important, orthogonal frequency division multiplexing/multiple access (OFDM/OFDMA) has become the most likely option for the next generation of wireless networks. Cooperative OFDM has gained a lot of attention from researchers recently [7]–[12], especially focusing on resource (subcarrier and power) allocation. As proved several times now, resource allocation is particularly important in OFDM-based systems. For cooperative networks, this includes the pairing of source and relay subcarriers, imposing selection on each subcarrier. The authors of [13] found upper and lower bounds on the outage and ergodic capacity of a three node relay system. In [14] the authors investigate resource allocation in a special case of Gaussian relay channel and show the considerable performance gains available by optimizing the resource allocation.

Although a lot of work has been done for relaying and resource allocation, the number of research works on multi-source, OFDMA, is limited. Li and Liu studied the capacity of OFDM-based relay networks for both amplify-and-forward (AF) and DF strategies [7] and the problem of maximizing the sum rate with fairness constraints in a multiple-source multiple-relay network using a graph theoretical approach [8]. In [9] Ng and Yu solve the general optimization problem for power allocation and relay selection as well as finding the best strategy for AF relaying or DF) in a cellular OFDM network. They have approached the problem defining a set of pricing variables and their basic assumption on the network nodes is that each pair of source-destination and possibly a relay are using their own specific frequency tones. Han *et al.* [11] investigate resource allocation in OFDMA networks using AF. In their work they assume at most one relay may help a user by assigning a portion of its subcarriers to that of the user being helped. In [12], authors develop the optimal resource allocation for a OFDMA system using DF with multiple source nodes and a single destination. The source nodes may allocate a portion of their subcarriers for relaying other users' messages. The authors find an allocation for networks with multiple sources.

All the works mentioned so far assume that if a source is helped, all subcarriers receive help. The general assumption is that the number of relays is greater than or equal to the number of sources. However, a relay may only be able to devote a *fraction* of its resources - or there may not be enough relays to go around. This entails using *fractional cooperation* wherein only some, but not all, of the subcarriers of a specific source are relayed. The concept of fractional cooperation was developed in the context of error control coding in [15]. A chosen fraction of the source's data is incorporated into the relay data and encoded for retransmission. With fractional cooperation, selection can be extended to choosing multiple relays each contributing a fraction of the source data [16]. Then the a random number of bits from this message are chosen, re-encoded and then transmitted to the destination. This fact makes fractional cooperation promising for networks with a relatively large number of source nodes.

In this paper we consider fractional cooperation based on the subcarriers in an OFDM block. Our work is most similar to the work in [12] wherein the power required to achieve a set of target rates is investigated. We take a different tack here, focusing on a mesh network that is rate, not power, limited. The system comprises N_S source nodes being helped with

$N_R (< N_S)$ relays all using N subcarriers. The relays can therefore provide help to only $N_R \times N$ of $N_S \times N$ source subcarriers. We wish to maximize the minimum rate across all N_S sources. The contributions here are: (i) we provide an upper bound on the achievable max-min rate allowing for subcarrier permutation, i.e., any relay subcarrier can help any source subcarrier, (ii) present a simple and efficient algorithm to match subcarriers and allocate power with performance very close to the upper bound (iii) we show that, in general, achieving the maximum rate requires the number of relays to be significantly lower than the number of sources.

The rest of the paper is structured as follows. Section II, develops the system model for the multiple-source OFDM-based network under consideration. In Section III the issue of resource allocation on a per-subcarrier basis is investigated and an upper bound on the achievable rate as well as an efficient algorithm are developed. Section IV presents the results of simulations that illustrate the workings of the theory presented. Finally, section V concludes this paper.

II. System Model

The system under consideration is a network of N_S source nodes attempting to transmit their messages to a single destination. There are N_R dedicated relay nodes helping this communication. In general $N_R < N_S$, though this is not necessary. The decoding strategy is decode-and-forward and we are assuming the relays are fully capable of decoding the received messages from all the sources, i.e., the channel from the source nodes to the relay nodes is perfect. Essentially we are making the fairly common assumption [9], [12] that all relays can decode all source messages. This should be especially true for dedicated relays. The source-destination and relay-destination channels are Rayleigh. Fig. 1 illustrates the network described. All transmissions use N subcarriers. The channel link from the source and relay nodes to the destination node is a multi-tap Rayleigh fading channel. Assuming that a sufficient guard interval is used, the overall channel model for each link is reduced to N parallel Gaussian channels.

Communications occurs over two stages. In the fist stage, the source nodes take turns, over N_S OFDM blocks, in sending their data to the destination and the relays. During this time, the relay nodes and the destination node listen and store the received signals from all the sources. Since the relays have perfect links from the source nodes they can decode the messages at this stage. In the second stage the relays, in turn, forward data to the destination using N_R OFDM blocks.

To optimize resources in the second stage, we assume that the destination has knowledge of all relevant receiver channel state information (CSI), i.e., the source-destination and relay-destination links. In the following stage, the relays decide on their own OFDM symbols to transmit to the destination node. Each relay looks for some N subcarriers to re-transmit out of all the $N \times N_S$ subcarriers received from all the sources. Note that since the relays each transmit within their own time-slot, the optimal power allocation is obtained via waterfilling. The

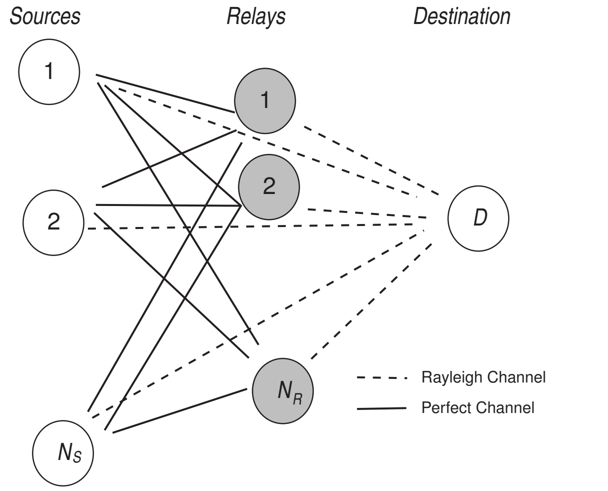

Fig. 1. Cooperative multi-source, multi-relay, and single-destination network

key optimization here is, therefore, the allocation of $N_R \times N$ relay subcarriers to $N_S \times N$ source subcarriers.

Let $p_{i,j}$ denote the power allocated by source i to its j^{th} subcarrier and $q_{i,j}$ denote the power allocated by relay i to its j^{th} subcarrier. In addition, $h^s_{i,j}$ and $h^r_{i,j}$ are the channel gains for the j^{th} subcarrier from node or relay i respectively. Since the channel is assumed to be Rayleigh, $h^s_{i,j}$ and $h^r_{i,j}$ are complex Gaussian random variables.

The rate R_i from source node i to the destination node is given by

$$R_i = \frac{1}{N_R + N_S} \sum_{j=1}^{N} \log_2 \left(1 + |h^s_{i,j}|^2 p_{i,j} + X_{i,j}/\sigma^2\right) \quad (1)$$
$$X_{i,j} = \sum_{(m,n) \in S_{i,j}} q_{m,n} |h^r_{m,n}|^2$$

where the term $X_{i,j}$ represents the contribution of relay nodes to the j^{th} subcarrier of source i. Here σ^2 is the variance of noise and $S_{i,j}$ is the set of subcarriers which are retransmitting subcarrier j of source i. The objective is to maximize the minimum rate among all the source nodes by assigning relay subcarriers to the source subcarriers, i.e., by partitioning the set of all relay subcarriers into proper sub-sets of $S_{i,j}$ in order to maximize $[\min_i R_i]$.

The next section presents the formulation of the optimization problem, an upper bound to the solution, and a sub-optimal algorithm for the subcarrier assignment.

III. Subcarrier Selection

Based on the system model described above and (1) we can formulate the optimization problem which achieves the max-min rate in this cooperative network. The problem can be formalized as follows. We define the binary variable $L_{i,j,k,l}$ to denote subcarrier allocation. If $L_{i,j,k,l} = 1$ subcarrier j of source node i is relayed by subcarrier l of relay node k. Note that given k and l, only one of $L_{i,j,k,l}$ can be non-zero for $1 \leq i \leq N_S$ and $1 \leq j \leq N$. Equation set (1) can be rewritten as

$$R_i = \frac{1}{N_R + N_S} \sum_{j=1}^{N} \log_2 \left(1 + \frac{|h^s_{i,j}|^2 p_{i,j} + X_{i,j}}{\sigma^2} \right) \quad (2)$$

$$X_{i,j} = \sum_{k=1}^{N_R} \sum_{l=1}^{N} \left(L_{i,j,k,l} q_{k,l} |h^r_{k,l}|^2 \right)$$

Using (2) the optimization problem can we described as:

$$\max_{\{L_{i,j,k,l}\}} R_{min} \quad (3)$$

subject to :

$$R_i \geq R_{min} \quad \forall \quad i \in 1, 2, ..., N_S$$

$$\sum_{i=1}^{N_S} \sum_{j=1}^{N} L_{i,j,k,l} = 1 \quad \forall \quad 1 \leq k \leq N_R, 1 \leq l \leq N$$

$$L_{i,j,k,l} \in \{0, 1\}$$

This problem is a general combinatorial 0-1 programming problem and unfortunately, there is no known efficient method to solve it. Therefore we have to look for sub-optimal solutions which are hopefully efficient and more practical. Next we will propose an algorithm and also compare it with an upper bound on the solution of (3).

A. Upper bound

One way to achieve an upper bound is to simply ignore the selection condition in (3) and assume $L_{i,j,k,l}$ as a real valued variable in $[0, 1]$. Here we take a more intuitive approach in finding the upper bound. Looking back at the main optimization problem, we are trying to divide the relay subcarriers into N_S subsets in order to achieve the maximum value possible for the minimum rate among all the source nodes. Then when a set of subcarriers is assigned to a certain node, letting $X_{i,j} = \sum_{(m,n)\in S_{i,j}} q_{m,n} |h^r_{m,n}|^2$, the upper bound can be found through the following optimization problem:

$$\max_{\{X_{i,j}, R_{min}\}} R_{min}$$

subject to :

$$R_i = \frac{1}{N_R + N_S} \sum_{j=1}^{N} \log_2 \left(1 + \frac{|h^s_{i,j}|^2 p_{i,j} + X_{i,j}}{\sigma^2} \right) \geq R_{min}$$

$$X_{i,j} \geq 0$$

$$\sum_{i=1}^{N_S} \sum_{j=1}^{N} X_{i,j} \leq \sum_{k=1}^{N_R} \sum_{l=1}^{N} q_{k,l} |h^r_{k,l}|^2 \quad (4)$$

Essentially, the upper bound relaxes the constraint on $H_{i,j}$ stated before this equation. The optimization in (4) is over only $N_S \times N + 1$ variables whereas in (3) this number was $N_S \times N_R \times N^2 + 1$. Furthermore, this is a convex optimization problem. Therefore, this optimization problem for finding the upper bound is numerically more practical than that of the relaxed optimization problem (3).

If we further relax the condition on positiveness of $H_{i,j}$ in (4), we can find a looser upper bound which can actually be derived analytically. The Lagrangian for the problem (4) is:

$$L = R_{min} + \sum_{i=1}^{N_S} (R - R_i)\lambda_i + \lambda \left(\sum_{i,j} X_{i,j} - H_{total} \right) \quad (5)$$

where the positive value, X_{total} is defined as:

$$X_{total} = \sum_{k=1}^{N_R} \sum_{l=1}^{N} q_{k,l} |h^R{}_{k,l}|^2$$

At the optimal point, the gradient of the Lagrangian with respect to the problem variables is zero. It follows:

$$\frac{\partial L}{\partial R_{min}} = 0 \Rightarrow \sum_{i=1}^{N_S} \lambda_i = 1$$

$$\frac{\partial L}{\partial X_{i,j}} = 0 \Rightarrow \frac{\lambda_i.\sigma^2}{\sigma^2 + C_{i,j} + X_{i,j}} + \lambda = 0 \quad (6)$$

where

$$C_{i,j} = |h^s{}_{i,j}|^2.p_{i,j}. \quad (7)$$

Applying (6) for two different subcarriers of the same source node we obtain:

$$\frac{\lambda_i \sigma^2}{\sigma^2 + C_{i,j} + X_{i,j}} = \frac{\lambda_i.\sigma^2}{\sigma^2 + C_{i,k} + X_{i,k}}$$

$$\Rightarrow C_{i,j} + X_{i,j} = C_{i,k} + H_{i,k} \quad (8)$$

which suggests that the solution to the problem follows watefilling. Therefore, the optimal result will be such that the level of $C_{i,j} + H_{i,j}$ is equal for all subcarriers of a source node, and consequently equal to that of subcarriers of other nodes. It follows:

$$L_{avg} = C_{i,j} + X_{i,j}$$

$$\Rightarrow N_S N L_{avg} = \sum_{i,j} (C_{i,j} + X_{i,j})$$

$$L_{avg} = \frac{\sum_{i,j} C_{i,j} + X_{total}}{N_S.N}$$

$$\Rightarrow R_{min,ub}\left(\sigma^2\right) = \frac{N}{2} \log \left(1 + \frac{L_{avg}}{\sigma^2} \right) \quad (9)$$

At the threshold point where relaying and non-relaying systems are equal we have:

$$R_{min,ub}(\sigma^2) = \min_i R_i(\sigma^2) \quad (10)$$

Now the solution to (10) is a lower bound (upper bound) on the variance of noise (signal-to-noise ratio : SNR) at which relaying is not useful anymore. In Section IV, we compare the upper bound, i.e., solution to (4) and the simplified algorithm given below in Section III-B for different scenarios.

B. Selection Algorithm

The original optimization problem described in (3) is not solvable in reasonable time. While the upper bound can be obtained for small values of N_S, N_R and N, it too is intractable for reasonable values. We therefore need to look for a practical sub-optimal algorithm. Instead of looking at all the subcarrier of all the relays at the same time and trying to find the optimum mapping method out of the $(N_S \times N)^{N_R \times N}$ possible ways, we try to assign some rules based on which a reasonably close-to-optimal solution result is achieved. We describe below a fairly straightforward and intuitive approach.

We define a subcarrier with large $q_{i,j}|h^r_{i,j}|^2$ or $p_{i,j}|h^s S_{i,j}|^2$ term, for a relay or a source respectively, as a strong subcarrier. A subcarrier for which these terms are small is called a weak subcarrier. We also refer to the mentioned quantities as subcarrier values. Clearly the goal is to help the weak subcarriers and not the strong ones.

In order to make the assignment problem simple we divide it into a sequential assignment procedure. Our goal is to increase the minimum rate among the source nodes at each step. Initially, when no subcarriers have already been assigned, the node with the least rate will be assigned a subcarrier. Our second goal is to make the largest possible change in rate at each step. As we know the rate function is a concave function of the subcarrier channel gains and allocated power. Therefore, helping a weaker subcarrier results in greater increase in the overall rate. Hence, the strongest relay subcarrier is be assigned to the weakest subcarrier of the node with the least rate. After this assignment, the rate value for the weakest node as well as its weakest subcarrier value (which is now the previous value plus the value taken from the strongest relay subcarrier) is updated. The rest of the assignment is done similarly. In short the algorithm works based on the following rules:

1) Look for the node with the least rate so far.
2) Assign the largest subcarrier among all the relays which is not previously assigned to any other source subcarrier to the weakest subcarrier of the chosen node.

Steps 1 and 2 are repeated until all the subcarriers of all the relays are assigned. As it is obvious the complexity of this method is hugely reduced than that of the solution to (3) or even (4). The complexity is essentially that of sorting $N_S \times N$ numbers. This method is not optimal but our numerical results later will show that the result achieved by this method is close to optimal.

IV. Numerical Results

In this section we compare the performance of the algorithm presented with that of (4) the proposed upper bound on the solution to (3). We set $N = 32$ subcarriers. The channel for all the links is a six-tap Rayleigh fading channel. For a given number of source and relay nodes and a fixed SNR, we run average the results over 100 different channel realizations. All channels have equal average power. Finally, all schemes are compared in terms of the achievable max-min rate.

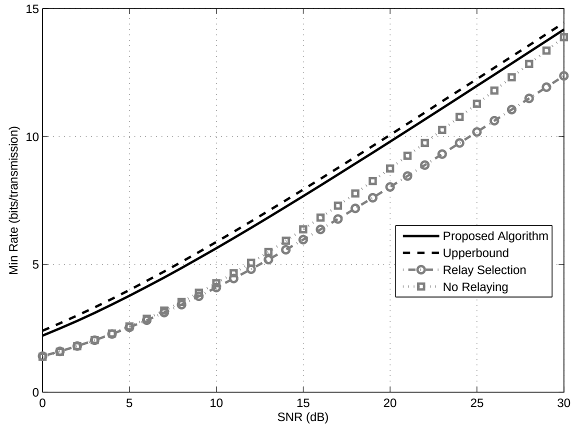

Fig. 2. Comparison of the upper bound, proposed algorithm for relay selection and no relaying.

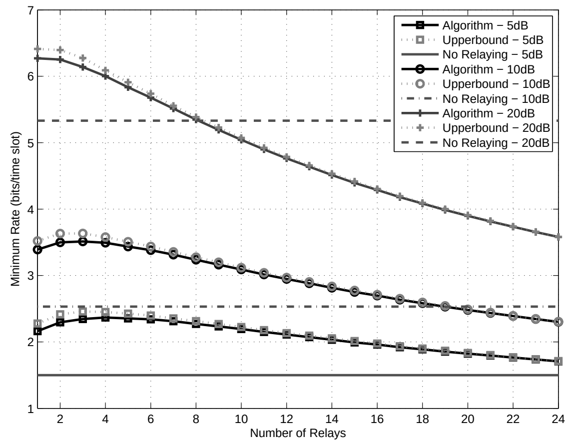

Fig. 3. The optimal number of relays for a given network and given SNR.

The first example uses $N_s = 10$ source nodes and $N_R = 2$ relay nodes. Figure 2 presents performance of the proposed algorithm, the upper bound, relaying based on selection (without power allocation), and no relaying for a wide range of SNR. Since the number of relays is limited, and only the two nodes with the minimum rate need relaying, the optimum selection pattern is simply found through a brute force search. As the figure suggests, the performance of simple algorithm presented in Section III-B is extremely close to that of the upper bound. Essentially, the upper bound is extremely tight and the algorithm extremely efficient..

Figure 3 shows an interesting result obtained from our work. Note that in this figure, N_R varies from 1 to 25 with $N_s = 16$. While in the theory presented above was designed for $N_R < N_S$, this is not a requirement. This figure shows that for a given number of source nodes and SNR, there is a certain number of relays which on average results in highest max-min rate among the source nodes. Interestingly, this number is lower than the number of sources,

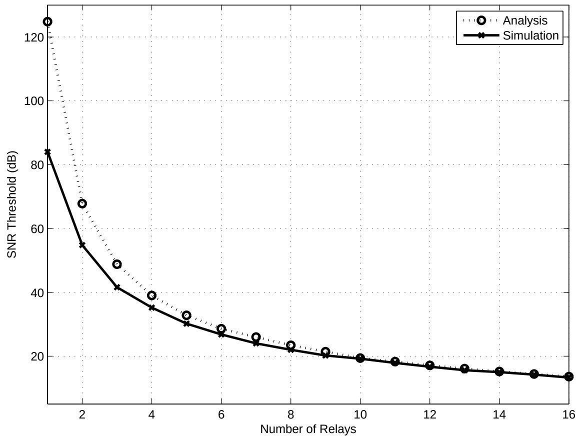

Fig. 4. The threshold of SNR at which the relayed system performs as well as the non-relayed system vs. number of relay nodes.

i.e., to maximize the minimum rate, in general we must pick $N_R < N_S$. This is because adding relays carries the overhead of requiring additional time slots for transmission. For the stronger subcarriers, the improvement in rate (the inside-the-log factor) does not compensate for this overhead (the pre-log factor). This is consistent with the fact that the optimum number of relays is is lower for networks with high SNR and higher for networks with low SNR.

As described in Section III-A, for some given networks with sufficiently high SNR, on average it is better not to relay at all. In Fig. 2 there is a certain SNR at which the relayed system (upper bound on rate) and the non-relayed system perform similarly. Therefore for the higher SNR levels it is better not to relay in this system. Also in Fig. 3, for each given SNR, there is a certain number of relays at which the upper bound crosses the rate obtained from non-relayed system and increasing the number of relays more will result into loss of data rate.

For a given network, the SNR at which the relayed and non-relayed systems become equal (using the max-min rate metric) is approximately found by solving (10). Fig. 4 depicts the validity of the analysis. Here simulation over different channel instances has been run for a network of 16 source nodes and different number of relay nodes. For each case the threshold has been found both through simulation and analysis based on (10). The threshold is almost inversely proportional to the number of relay nodes in the network. Also, the graph suggests that the approximation made in (10) is good and for higher number of relays is very close to the actual value.

V. CONCLUSIONS

This paper discusses subcarrier allocation for maximizing the minimum rate in a multiple-source, multiple-relay, and single destination cooperative network with OFDM as the underlying transmission scheme. Subcarriers of a relay may be assigned to different source nodes which is suggestive of fractional cooperation. Through fractional relaying it is shown that better performance can be achieved than that of selection method. The optimal subcarrier assignment problem is formulated and an upper bound to the solution in the form of a convex optimization problem is presented. In addition, a low-complexity and efficient algorithm for sub-optimal subcarrier assignment is discussed. Numerical results show that the presented upper-bound is very close to the results obtained from the algorithm which indicates the efficiency of the algorithm, as well as the tightness of the upper-bound. In addition it is observed that subcarrier level selection greatly outperforms the conventional relay selection methods.

Our results also show that for a fixed SNR, there is a certain number of relays which maximizes the overall rate, or in other words, for a fixed network, there is a SNR threshold for which and the SNR values above the non-relayed network outperforms the relayed one. An upper bound on that threshold is also presented through solution of a non-linear equation.

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