Blind Channel Estimation for Orthogonal STBC in MISO Systems

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Abstract

This paper presents a closed-form blind channel estimation scheme for orthogonal space-time block codes in multiple-input single-output (MISO) systems, with specific focus on Alamouti's code for two transmit antennas. The channel matrix is estimated from the eigenvalue decomposition of the fourth-order cumulant matrix of the received signal. Unlike previous blind estimation schemes for MISO systems, the proposed algorithm is tested with block and slowly fading channels. The proposed scheme performs very well in both cases. A single pilot-tuple is required to correctly assign the estimated to the actual channels and to resolve the sign ambiguity common to all blind estimators. It is shown that this scheme outperforms the only other available blind channel estimation scheme for this scenario. To achieve good performance in terms of bit error rate, $100-300$ sample points are sufficient to provide accurate channel estimates. The main disadvantage of the proposed scheme is the complexity associated with estimation of fourth-order cumulants. This complexity is reduced by exploiting the symmetry inherent in the cumulant matrix.

I. Introduction

The advantages of using multiple transmit and/or receive antennas along with space-time coding have been extensively studied [1]–[3] and are now well accepted. In particular, the orthogonal space-time block codes (O-STBC) [2], specifically Alamouti's code [3], have been shown to be very attractive in terms of providing full diversity with linear decoding complexity. However, the performance of these codes depends on accurate knowledge of the channels between the transmit and receive antennas. The importance of channel information to space-time coding has motivated investigation of channel estimation for multiple-input-multiple output (MIMO) systems [4]. As with the single-input-single-output (SISO) case, training, blind and semi-blind techniques have been proposed.

Focusing on MIMO systems using STBC, one class of approaches exploits the structure of the space-time codes to enable channel estimation [5]–[14]. Budianu and Tong [5] and Larsson *et al.* [6] present training based schemes for the orthogonal codes of Alamouti [3] and Tarokh [2]. Training bits, however, reduce effective throughput and such schemes are inappropriate for systems where bandwidth is scarce. By restricting themselves to real signals and transmit diversity order, Ammar and Ding estimate channels for STBC from the null space of the received signal [7]. Swindlehurst and Leus present a scheme for blind channel estimation with a generalized set of space-time codes [8]. Larsson *et al.* [10] present a blind optimal, in maximum likelihood (ML) sense, scheme for channel estimation. Ma and co-authors [11]–[13] simplify the problem by exploiting the O-STBC structure and semi-definite relaxation [11] or sphere decoding [12]. However, the complexity of ML decoding remains. Similarly, Shahbazpanahi *et al.* present a closed form channel estimate used for ML decoding of transmitted symbols [14].

A significant problem with most of these blind approaches is that they require the number of receive antennas to be greater than or equal to the number of transmit antennas [7], [8]. In [10], [11] this requirement is not explicit, but all numerical examples use such a scenario. In a large part, space-time coding is designed for transmit diversity in the downlink. Assuming multiple receive elements on a mobile device as at the base station may not be realistic. Stoica and Ganesan [9] present an iterative algorithm for blind channel estimation which does not place restrictions on the input signal or on the number of antennas. However, the algorithm is very sensitive to initialization and the authors acknowledge the results to be unsatisfactory; they improve the algorithm through semi-blind and training-based estimation. The work in [14] develops a blind channel estimator for O-STBC which requires a precoder for the transmitted data. To our knowledge, this is the only blind channel estimation algorithm appropriate to the scenario discussed here. We show in this paper that at the price of increased complexity, the performance of this scheme could be improved through the use of higher order cumulants.

This paper presents an effective blind channel estimation technique for downlink systems using a class of O-STBC and only one receive antenna. The specific focus, and the most important application, is the Alamouti code for two transmit antennas. The technique builds on the work presented by Ding and Liang [15], who introduce a special form of the cumulant matrix to estimate a finite impulse response SISO channel. We show that multiple channels can be estimated from the eigenvectors of the cumulant matrix up to a single sign ambiguity. A single known symbol inserted into the transmitted data stream is shown to be sufficient to resolve this ambiguity.

This paper is structured as follows. Section II presents the data model used in this paper based on a single receive antenna and the definition of cumulants used later. Section III presents the theory of the proposed approach for MISO systems. Section IV discusses implementation issues while Section V presents simulations to illustrate the performance of the proposed scheme. Section VI concludes this work.

II. PRELIMINARIES

A. Data and Channel Model

Consider a system with $L > 1$ transmit antennas and one receive antenna. A block of N complex data symbols, $\mathbf{s}$, is encoded over K time slots using the generalized orthogonal space-time codes of [2]. The channel is modelled as flat and Rayleigh. The assumptions used in this work are:

1) source symbols are zero mean, independent, identically distributed (i.i.d.) random variables with non-zero fourth-order kurtosis (defined later), γ_4,
2) receiver noise is additive, white and Gaussian, and
3) the channel matrix is constant over the K time slots.

Under these assumptions, the length-K receive data vector $\mathbf{r}$ can be written in the most general form as

$$\mathbf{r} \quad = \quad \mathbf{G}_r\mathbf{h} + j\mathbf{G}_i\mathbf{h} + \mathbf{w}, \tag{1}$$

where $\mathbf{G}_r$ and $\mathbf{G}_i$ are related below to $\mathbf{s}_r$ and $\mathbf{s}_i$, the real and imaginary parts of the transmitted signal $\mathbf{s}$ respectively, $\mathbf{h} = [h_1, h_2, \dots, h_L]^T$ denotes the L channels between the transmit antennas and receive antenna and ^{T} the transpose of a matrix. The vector $\mathbf{w}$ represents the additive white, Gaussian, receiver noise. The $K \times L$ matrices, $\mathbf{G}_r$ and $\mathbf{G}_i$, dependent on the particular code used, represent the encoding of the real and imaginary parts of the transmit signal vector. They are formed from real orthogonal matrices $\mathbf{X}_n$ and $\mathbf{Y}_n$, $n = 1, \dots, N$,

$$\mathbf{G}_r = \sum_{n=1}^{N} \mathbf{X}_n s_{rn}, \qquad \mathbf{G}_i = \sum_{n=1}^{N} \mathbf{Y}_n s_{in}. \tag{2}$$

Using O-STBC, the sets of matrices $\{\mathbf{X}_n\}$ and $\{\mathbf{Y}_n\}$ satisfy the following properties [6]:

$$\begin{aligned} {\mathbf{X}_n}^T\mathbf{X}_n &= \mathbf{I}_L, \qquad {\mathbf{Y}_n}^T\mathbf{Y}_n = \mathbf{I}_L, \qquad \forall n, && (3)\\ {\mathbf{X}_n}^T\mathbf{X}_m &= -{\mathbf{X}_m}^T\mathbf{X}_n, \quad {\mathbf{Y}_n}^T\mathbf{Y}_m = -{\mathbf{Y}_m}^T\mathbf{Y}_n, \quad n \neq m, && (4)\\ {\mathbf{X}_n}^T\mathbf{Y}_m &= {\mathbf{Y}_m}^T\mathbf{X}_n, \quad n \neq m, && (5)\end{aligned}$$

As in [14], the real and imaginary parts of the received signal can be processed independently. We can thus rewrite (1) as

$$\underline{\mathbf{r}} = \mathbf{H_c}\underline{\mathbf{s}} + \underline{\mathbf{w}}, \tag{6}$$

where the underline operator is used to denote the stacking operations, i.e.,

$$\underline{\mathbf{r}} = \begin{bmatrix} \Re(\mathbf{r}) \\ \Im(\mathbf{r}) \end{bmatrix}, \tag{7}$$

where $\Re(x)$ and $\Im(x)$ represent the real and imaginary parts of x respectively. The real channel matrix $\mathbf{H_c}$ is formed from the matrices $\{\mathbf{X}_n\}$ and $\{\mathbf{Y}_n\}$ and the channel vector $\mathbf{h}$:

$$\mathbf{H}_c = \begin{bmatrix} \underline{\mathbf{X_1 h}} & \dots & \underline{\mathbf{X_N h}} & \underline{\mathbf{Y_1 h}} & \dots & \underline{\mathbf{Y_N h}} \end{bmatrix}. \tag{8}$$

A important property of this matrix $\mathbf{H}_c$ is that its rows and columns are orthogonal [14], i.e.,

$$\mathbf{H_c}^T\mathbf{H_c} = ||\mathbf{h}||_2^2\mathbf{I}_{2N}, \tag{9}$$

where $\|\ \mathbf{h}\ \|_2^2$ denotes the 2-norm of the channel vector, and $\mathbf{I}_{2N}$ is the $2N \times 2N$ identity matrix. For example, in the Alamouti scheme, $\mathbf{h} = [h_1,\, h_2]^T = [h_{1r} + jh_{1i},\, h_{2r} + jh_{2i}]^T$ and the matrix $\mathbf{H_c}$ is

$$\mathbf{H}_c = \begin{bmatrix} h_{1r} & h_{2r} & -h_{1i} & -h_{2i} \\ -h_{2r} & h_{1r} & -h_{2i} & h_{1i} \\ h_{1i} & h_{2i} & h_{1r} & h_{2r} \\ -h_{2i} & h_{1i} & h_{2r} & -h_{1r} \end{bmatrix}, \tag{10}$$

and $\mathbf{H}_c^H\mathbf{H}_c = \mathbf{H}_c\mathbf{H}_c^H = \left(|h_1|^2 + |h_2|^2\right)\mathbf{I}_2$. As in [14], this property is essential to the estimation algorithm presented in this paper.

B. Fourth Order Cumulants

The channel estimation algorithm in this paper is based on fourth-order cumulants. The required definitions are presented below.

Let $c_q(x_1, x_2, \ldots, x_q)$ represent the q^{th} order joint cumulant and $m_q(x_1, x_2, \ldots, x_q)$ the q^{th} order moment of q random variables $(x_1, x_2, \ldots, x_q)$. The q^{th} order moment is defined as

$$m_q(x_1, x_2, \ldots, x_q) = E\{x_1 x_2 \ldots x_q\}, \tag{11}$$

where $E\{\cdot\}$ represents statistical expectation. For the zero-mean random variables often used in practice, the cumulants of order 2 and 4 are defined as

$$\begin{aligned} c_2(x_1, x_2) &= m_2(x_1, x_2), && (12) \\ c_4(x_1, x_2, x_3, x_4) &= m_4(x_1, x_2, x_3, x_4) - m_2(x_1, x_2)m_2(x_3, x_4) \\ &\quad - m_2(x_1, x_3)m_2(x_2, x_4) - m_2(x_1, x_4)m_2(x_2, x_3). && (13) \end{aligned}$$

The quantity γ_q is defined as $\gamma_q = c_q(x, x^*, \ldots x, x^*)$. The variance and kurtosis of x are, respectively,

$$\begin{aligned} \gamma_2 &= c_2(x, x^*) && (14) \\ \gamma_4 &= c_4(x, x^*, x, x^*) = E\{|x|^4\} - 2\left[E\{|x|^2\}\right]^2 - E\{(x)^2\}E\{(x^*)^2\}. && (15) \end{aligned}$$

Note that the cumulants are linear in each variable and that the 4^{th}-order cumulant of jointly Gaussian random variables, such as white noise, is zero [16]. Also, for a white process, $\{x_n\}$,

$$c_4(x_n, x_{n-n_1}, x_{n-n_2}, x_{n-n_3}) = \gamma_4 \delta(n_1)\delta(n_2)\delta(n_3). \tag{16}$$

III. Channel Estimation Using Fourth Order Cumulants

A. The Estimation Algorithm

As in [15], define the joint cumulant matrix of the vector $\underline{\mathbf{r}}$ as

$$\mathbf{C}_4^{[k]} = c_4(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T, r_k, r_k) \quad k = 1, \ldots, 2K, \tag{17}$$

i.e., $\mathbf{C}_4^{[k]}(i,j) = c_4(r_i, r_j, r_k, r_k)$.

Proposition: Each eigenvector of the cumulant matrix $\mathbf{C}_4^{[k]}$ is an unknown permutation of a scaled column of the channel matrix, $\mathbf{H}_c$.

Proof:

¿From the definition in (13), the joint cumulant is linear in each of its arguments. As in [15], the cumulant matrix can therefore be decomposed as

$$\begin{aligned} \mathbf{C}_4^{[k]} &= c_4(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T, r_k, r_k) = c_4(\mathbf{H}_c\mathbf{s}, \mathbf{s}^T\mathbf{H}_c^T, r_k, r_k) \\ &= \mathbf{H}_c c_4(\underline{\mathbf{s}}, \underline{\mathbf{s}}^T, r_k, r_k)\mathbf{H}_c^T = \mathbf{H}_c\mathbf{B}_k\mathbf{H}_c^T. \end{aligned} \quad (18)$$

The entry in row i and column j of the inner matrix, $\mathbf{B}_k$, is given by $c_4(s_i, s_j, r_k, r_k)$. From (16), this term is non-zero only for $i = j$, i.e., $\mathbf{B}_k$ is diagonal. Therefore, using (9), the fourth-order cumulant matrix can be written as [15]

$$\mathbf{C}_4^{[k]} = \gamma_4\mathbf{H}_c\mathbf{F}_k\mathbf{H}_c{}^T. \quad (19)$$

From (15), $\gamma_4 = -\frac{P^2}{2}$, where P is the total transmitted power. The structure of matrix $2N \times 2N$ diagonal matrix $\mathbf{F}_k$ depends on the constellation used. For complex constellations, it can be written as

$$\mathbf{F}_k = \operatorname{diag}\left(|h_{k1}|^2, |h_{k2}|^2, \ldots, |h_{k2N}|^2\right), \quad (20)$$

where $\mathbf{h}_k = [h_{k1}, h_{k2} \ldots h_{k2N}]$ is the k^{th} row of $\mathbf{H}_c$. In this case, the entries of this diagonal matrix, denoted with $f_j, j = 1, \ldots, 2N$, are a real and imaginary permutation of the L channels and $(N - L)$ zeros. With real constellations, the N last diagonal entries of $\mathbf{F}_k$ are zero:

$$\mathbf{F}_k = \operatorname{diag}\left(|h_{k1}|^2, |h_{k2}|^2, \ldots, |h_{kN}|^2, 0, \ldots 0\right). \quad (21)$$

The eigenvalues λ_j and eigenvectors $\mathbf{v}_j$ of the cumulant matrix, $\mathbf{C}_4^{[k]}$, can be determined using property (9)

$$\begin{aligned} \mathbf{C}_4^{[k]}\mathbf{v}_j = \lambda_j\mathbf{v}_j \Rightarrow \gamma_4\mathbf{H}_c\mathbf{F}_k\mathbf{H}_c{}^T\mathbf{v}_j - \lambda_j\mathbf{v}_j &= 0, \quad (22) \\ \Rightarrow \gamma_4\mathbf{H}_c{}^T\mathbf{H}_c\mathbf{F}_k\mathbf{H}_c{}^T\mathbf{v}_j - \mathbf{H}_c{}^T\lambda_j\mathbf{v}_j &= 0, \\ \left(\gamma_4||\mathbf{h}||_F^2\mathbf{F}_k - \lambda_j\mathbf{I}_{2N}\right)\mathbf{H}_c{}^T\mathbf{v}_j &= 0 \\ \Rightarrow \left(\gamma_4||\mathbf{h}||_F^2\mathbf{F}_k - \lambda_j\mathbf{I}_{2N}\right)\mathbf{H}_c{}^T\mathbf{v}_j &= 0, \quad (23) \end{aligned}$$

The eigenvalue $\lambda_j = \gamma_4||\mathbf{h}||_F^2 f_j$ reduces the rank of $\mathbf{F}_k$ by one and consequently $\mathbf{H}_c^H\mathbf{v}_j$ is proportional to $\mathbf{p}_j$, the j^{th} column of the size-$2N$ identity matrix, i.e.,

$$\mathbf{H_c}^T \mathbf{v}_j = \beta \mathbf{p}_j. \tag{24}$$

Substituting this result into (22) results in ,

$$\begin{aligned} \gamma_4 \mathbf{H_c} \mathbf{F}_k \beta \mathbf{p}_j &= \lambda_j \mathbf{v}_j, \\ \Rightarrow \mathbf{v}_j &= \frac{\beta \gamma_4 |h_{kj}|^2}{\lambda_j} \mathbf{H_c} \mathbf{p}_j. \end{aligned} \tag{25}$$

Thus for $|h_{kj}| \neq 0$, $\mathbf{p}_j$ extracts successive columns of the channel matrix $\mathbf{H}_c$ ■

The proof above is valid as long as not all channels are equal, i.e., $h_i = h_j, \forall i, j$ is not allowed. This ensures that the non-zero eigenvalues of the cumulant matrix are distinct and that (24) is sufficient and necessary. Since the probability of $h_1 = h_2 = \cdots = h_L$ is zero, the length-L channel vector can thus be recovered from an eigenvector of the cumulant matrix. In general, however, no information in the code identifies the permutation in which the eigenvectors are arranged: there is no way to assign each eigenvector to its corresponding column of the channel matrix. This problem can be solved by inserting pilot symbols, as is discussed in IV-A.

As in [14], this procedure leaves the channel estimate ambiguous up to a single multiplicative factor. This ambiguity is common to all blind channel estimation techniques [8] and may be resolved using a *single* pilot symbol inserted at the beginning of the data block. If required, as shown in Appendix I, the magnitude of the ambiguity can be resolved using the eigenvalues of the cumulant matrix. The remaining sign ambiguity can be easily resolved with a pilot symbol.

The cumulant matrix $\mathbf{C}_4^{[k]}$ has as its final two arguments r_k, corresponding to the k^{th} received symbol. Theoretically, any choice of k, $k = 1 \dots 2K$ provides the required channel estimate. In practice, due to noise and the finite data record used to estimate the cumulants, each choice of k in the matrix $\mathbf{C}_4^{[k]}$ provides a slightly different channel estimate. Clearly, at the expense of computation load, one could obtain better channel estimates by repeating the process for all valid k and averaging the results.

The steps of the proposed algorithm are therefore:

1) Using a block of data, estimate the cumulant matrix $\mathbf{C}_4^{[k]}$ for $k = 1$ using (13) and (17).
2) Perform an eigendecomposition of this matrix and select the principle eigenvector.
3) Use a pilot symbol to resolve the sign ambiguity and to assign the eigenvector to one of the columns

of $\tilde{\mathbf{H}}_c$. This procedure is described in Section IV-A.

4) If required, resolve the magnitude ambiguity using (33).

5) Repeat for $k = 2, \ldots 2K$.

The theory developed above focuses on the MISO case. With multiple receive antennas, the procedure above can be repeated at each element. If the number of transmitters is greater than the number of receivers, the procedure and the work in [14] above appear to be the only effective blind schemes available. However, as mentioned earlier, if there are at least as many receive as transmit antennas, several other blind channel estimation techniques have been proposed [5]–[8], [10], [11].

Before discussing implementation issues associated with our proposed algorithm, we note a significant difference from the estimation algorithm of [14]. The algorithm of [14] is based on the estimation of the covariance matrix that, in the case of most orthogonal codes designed for one receive antenna can be decomposed as

$$\mathbf{R} = \mathbf{H_c}\mathbf{\Lambda}_s\mathbf{H_c}^T + \frac{\sigma^2}{2}\mathbf{I}_{2N}, \tag{26}$$

where $\mathbf{\Lambda}_s$ is the diagonal covariance of the real matrix $\underline{\mathrm{s}}$. Using the property in (9), it is clear that $\mathbf{R}$ is a scaled identity matrix, and thus the only way to blindly estimate the channel in those cases is to precode the data, thus replacing $\mathbf{\Lambda}_s$ with $\mathbf{D}_{2N}\mathbf{\Lambda}_s$, where $\mathbf{D}_{2N}$ is the precoding diagonal matrix. This procedure leads to a covariance matrix with a similar structure as the cumulant matrix in (19) in this paper, with the main difference that the matrix $\mathbf{D}_{2N}\mathbf{\Lambda}_s$ is known a priori, while the matrix $\mathbf{F}_k$ is composed of the unknown channel powers.

Precoding is thus necessary to enable channel estimation with a covariance matrix; its use, however, does not introduce the ambiguity of the permutation, because the form of the precoding matrix is known a priori. Estimation using the cumulant matrix can be performed without precoding; it does, however, introduce the permutation ambiguity which must be resolved with the insertion of a pilot-tuple into a window of data. As expected, both schemes are ambiguous within a multiplicative constant.

IV. Implementation Issues

A. Ambiguity Resolution

The proposed channel estimation scheme extracts channel estimates from the eigenvector of the cumulant matrix. Each eigenvector is a permutation of a column of the channel matrix, $\mathbf{H}_c$. The scheme, however, does not identify the permutations. This ambiguity is similar to the ambiguity described by the authors in [17]. The O-STBC encodes N data symbols over K epochs and transmits the code over L antennas. Only for square STBC ($N = K = L$) schemes are the columns of the channel matrix the permutations of the channel vector $\mathbf{h} = [h_1, h_2, \ldots, h_L]$. This is the case for the Alamouti code, as well as for real square codes utilizing 4 and 8 antennas. This case is investigated in detail in this section. However, we begin with the simpler case of generalized rectangular block codes.

For such codes, the columns of the channel matrix, and thus the eigenvectors of the cumulant matrix, are augmented with $2(N - L)$ zeros. In the case of the full-rate real code for 3 antennas, for example, each column of the channel matrix is augmented with two zeros:

$$\mathbf{H}_c = \begin{bmatrix} h_{1r} & h_{2r} & h_{3r} & 0 \\ h_{2r} & -h_{1r} & 0 & -h_{3r} \\ h_{3r} & 0 & -h_{1r} & h_{2r} \\ 0 & h_{3r} & -h_{2r} & -h_{1r} \\ h_{1i} & h_{2i} & h_{3i} & 0 \\ h_{2i} & -h_{1i} & 0 & -h_{3i} \\ h_{3i} & 0 & -h_{1i} & h_{2i} \\ 0 & h_{3i} & -h_{2i} & -h_{1i} \end{bmatrix}. \tag{27}$$

The locations of the zero in the eigenvector can help resolve the permutation, i.e., identify which entry in the eigenvector corresponds to which channel. In the case of the real rectangular code shown, for example, the column number can be identified as $K - p_z + 1$, where p_z is the location of the first zero in the column. This is also true for the generalized complex O-STBC.

For a space-time code with $N = K = L$, such as Alamouti's scheme, the eigenvector provides estimates of the L channels. However, no information in the code itself allows a correct assignment between the estimated eigenvectors of the cumulant matrix and the columns of the channel matrix. A resolution of this problem therefore requires the use of one time slot for a single pilot transmission[1]. This ambiguity can be resolved by transmitting a single pilot symbol which extracts a chosen column from the channel matrix. Exploting the orthogonal property of the channel matrix, the resulting vector can be used to determine the corresponding column from the estimated channel matrix. This procedure is summarized below:

1) Transmit the known pilot symbol $\underline{\mathbf{s}} = \mathbf{p}_j$. As defined in Section III-A, $\mathbf{p}_j$ is the j^{th} column of the

[1]Note that all this ambiguity is independent of the phase ambiguity that impacts all blind schemes

size-$2N$ identity matrix. The received signal $\underline{\mathbf{r}}$ can thus be written as $\underline{\mathbf{r}} = \mathbf{h}_{c_j} + \mathbf{w}$, where $\mathbf{h}_{c_j}$ is the j^{th} column of the channel matrix $\mathbf{H}_c$.

2) Form the size-$2N$ row vector $\mathbf{m} = \mathbf{h}_{c_j}^T \tilde{\mathbf{H}}_c$.
3) Determine the location j of the maximum entry of $|\mathbf{m}|$. Due to the orthogonality of the columns of $\mathbf{H}_c$, j indicates the column of $\tilde{\mathbf{H}}_c$ corresponding to $\mathbf{h}_{c_j}$. The sign of the j^{th} entry of $\mathbf{m}$ also determines the sign ambiguity.

B. Cumulant Estimation

A crucial limiting factor in the implementation of the proposed algorithm is the complexity associated with estimating the fourth-order cumulant matrix. We present here schemes to limit the complexity of this estimation. Using (13), the fourth-order cumulant matrix is:

$$\begin{aligned}\mathbf{C}_4^{[k]} &= c_4\left(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T, r_k, r_k\right)\\ &= m_4(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T, r_k, r_k) - m_2(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T)m_2(r_k, r_k) - [m_2(\underline{\mathbf{r}}, r_k)m_2(\underline{\mathbf{r}}^T, r_k)]^2\\ &= m_4(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T, r_k, r_k) - m_2(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T)m_2(r_k, r_k) - [m_2(\underline{\mathbf{r}}, r_k)m_2^T(\underline{\mathbf{r}}, r_k)]^2.\end{aligned} \tag{28}$$

Note that $m_2(\underline{\mathbf{r}}, r_k)$ and $m_2(r_k, r_k)$ are the k^{th} column and diagonal entry of $m_2(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T)$, respectively. Thus to determine the cumulant matrix, only the following two terms need be calculated: $m_4(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T, r_k, r_k)$, and $m_2(\underline{\mathbf{r}}, \underline{\mathbf{r}}^T)$. Other efficient estimates of the fourth order cumulants are also possible [18].

C. Error Analysis

This section discusses the relationship between the normalized mean squared error (NMSE) and the number of sample points used to estimate the cumulant matrix. Given estimate $\hat{\mathbf{h}}$ of the true channel $\mathbf{h}$, the NMSE, is defined by

$$\text{NMSE} = \frac{\sum_{k=1}^{L} |h_k - \hat{h_k}|^2}{\sum_{k=1}^{L} |h_k|^2} = \frac{||\mathbf{h} - \hat{\mathbf{h}}||^2}{||\mathbf{h}||^2}, \tag{29}$$

where $||\cdot||$ refers to the 2-norm.

Proposition: For the case of the Alamouti code with $L = 2$ two transmit and one receive antenna, and a given channel $\mathbf{h} = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix}$, where $|h_0| \neq |h_1|$, the NMSE is approximated by

$$\text{NMSE} \approx \frac{|h_0|^4 + 3|h_0|^2|h_1|^2 + |h_1|^4}{B(|h_0|^2 - |h_1|)^2)^2}, \tag{30}$$

where B is the number of points used to calculate the cumulant estimate.

Proof: See Appendix II.

This expression is valid for the estimation scheme using regular cumulants, and without channel averaging, i.e. the channel estimate is obtained from only one cumulant matrix. As shown in Appendix II, the restriction that $|h_0| \neq |h_1|$ is due to an assumption, in the derivation, of distinct eigenvalues of the cumulant matrix. For identical channel magnitudes, the eigenvalues are identical (see Appendix I). Since the channels are modelled as Rayleigh, the event of equal channel magnitudes has zero probability.

V. Numerical Examples

In this section, the channel estimation algorithm presented in Section III is tested using simulations. All examples use BPSK for data modulation. Most examples are based on a slow, flat, Rayleigh block fading channel, i.e., the channel is constant over a block of data and changes independently from block to block. An important, and apparently unusual, test presented in Section V-B is based on a slow time-varying channel. The performance of the proposed scheme is evaluated using the normalized mean squared error and the resulting bit error rate (BER). The BER results are compared to that of a clairvoyant receiver using perfect knowledge of the channel.

A. Block Fading Channel

In the section, the channel to be estimated is held constant over each block of data. We focus on the important case of the Alamouti code for two transmit and one receive antenna. All data points in a window are used to estimate the cumulant matrix. As discussed in Section IV-A, the data in the first time slot of a window is assumed known to resolve any ambiguities. The channel changes independently from block to block. The results shown are averaged over 10^6 Monte Carlo simulations.

We first investigate the NMSE, defined in (29) between the true and estimated channels. The NMSE as a function of SNR is shown in Figure 1 for window sizes 50, 100, 300 and 500. As expected, the NMSE is a decreasing function of SNR. Although it is true that, in theory, cumulant estimates for white Gaussian

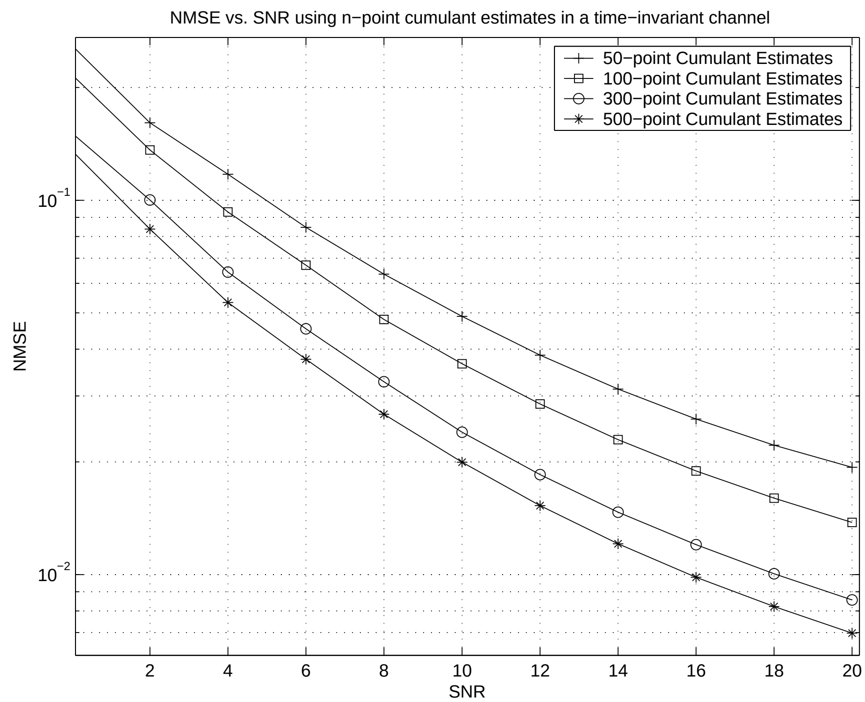

Fig. 1. Time Invariant Channel. NMSE vs. SNR using 50, 100, 300 and 500 point cumulant estimates.

noise are zero and the channel estimates should be insensitive to SNR, this occurs when the number of points used in the cumulant estimates is very large (we note here that 4^{th} order cumulants require a very large number of samples, of the order of thousands or more, for the estimated cumulants to converge to their true statistics. However, we do not require such a large number of samples to get fairly accurate channel estimates). When using a realistic number of samples, however, noise affects the accuracy of the estimation, and the channel estimates are sensitive to SNR. Figure 1 also indicates that, as expected, the NMSE is a function of window size: in a static environment, increasing the window size will improve the performance of the algorithm.

The proposition in Section IV-C is verified in Figure 2. The figure plots the NMSE of the channel estimate as a function of B, the number of samples used to estimate the cumulant matrix. The channel estimates in this plot were obtained from only one cumulant matrix; the NMSE performance is thus expectedly slightly worse than that obtained in Figure 1, where the channel estimate was obtained by averaging over $2K$ estimates from $2K$ cumulant matrices. Because the expression is only valid for distinct eigenvalues (which are identical for equivalent channel magnitudes), it is not accurate when the channel magnitudes become very close. For this reason, the difference between the channel magnitudes is restricted to 0.2 and above. The analytical expression approaches the simulated curve as B increases. This is as

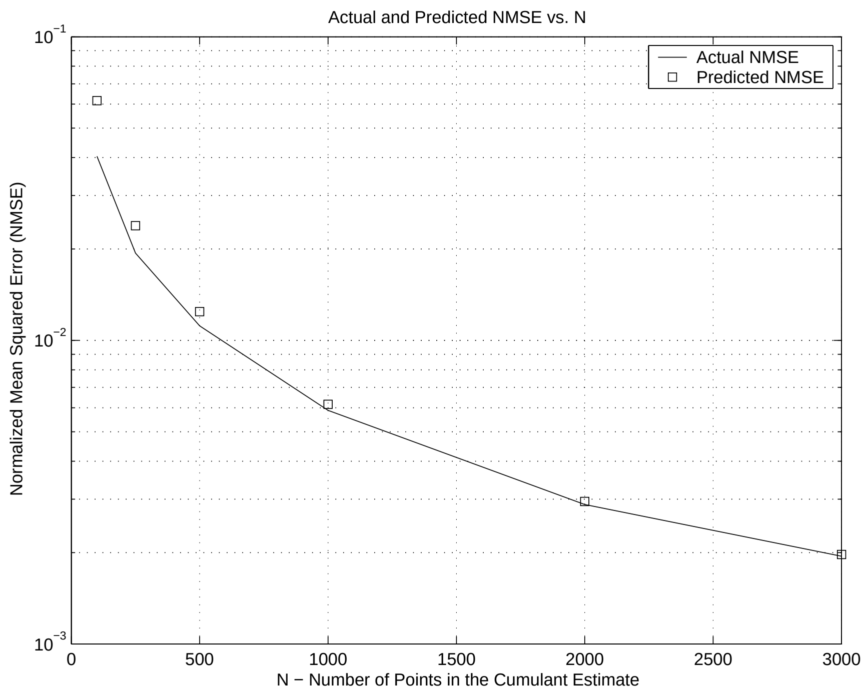

Fig. 2. Time-Varying Channel. NMSE vs. number of points used in cumulant estimates.

expected, since the approximations used to obtain (30) are based on a large B.

Channel estimation is one important step towards decoding the transmitted data. From a communication point of view, it is the BER that is finally important. The next plot, shown in Fig. 3 demonstrates the efficacy of the channel estimation in terms of the resulting BER. The BER is compared to that obtained by the clairvoyant receiver, which has knowledge of the true channel. As with the NMSE plots, the results are shown for window sizes 50, 100, 300 and 500.

Depending on the number of points used in cumulant estimates, the system BER, when using estimated channels, closely tracks that of the clairvoyant receiver. With 300 and 500-point cumulant estimates result in almost equivalent BER curves less than 1 dB from the Clairvoyant BER curve. As expected, in block fading channels, the performance of the algorithm can always be improved by increasing the number of points used in cumulant estimates.

In Figure 4, we compare our scheme to that presented in [14] in terms of BER. To obtain results for this algorithm, we use a precoding matrix for BPSK symbols $\mathcal{D} = \mathrm{diag}(\sqrt{0.4}, \sqrt{1.6}, 0, 0$. In [14], this matrix, used for QPSK symbols, is chosen in an ad hoc fashion; we thus did not optimize this matrix, but chose a similar one.

The figure demonstrates that our algorithm outperforms the scheme based on covariance estimates by 2 dB. The improved performance of our cumulant scheme is due to the decreased sensitivity to noise

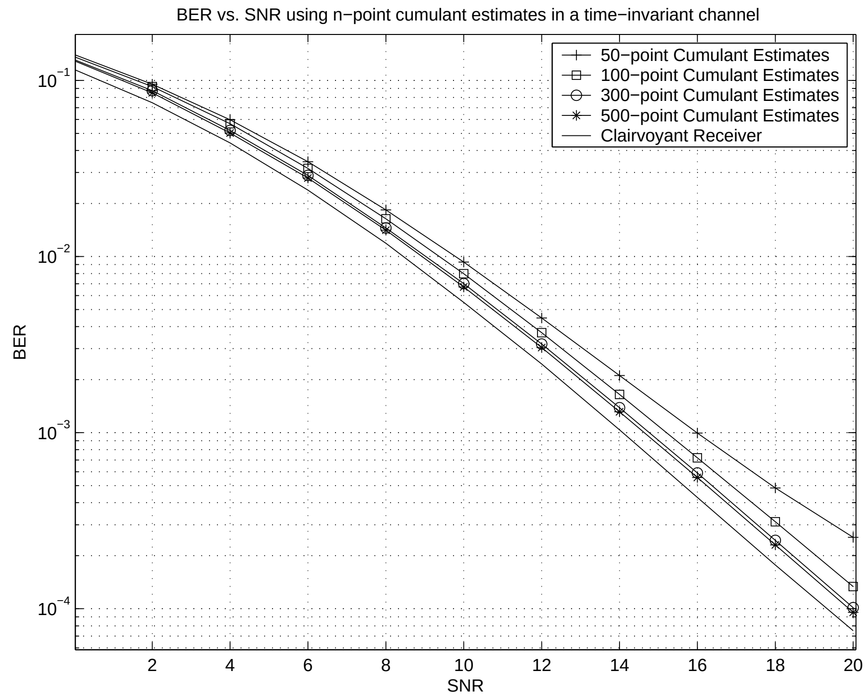

Fig. 3. Time Invariant Channel. BER vs. SNR using 50, 100, 300 and 500 point cumulant estimates.

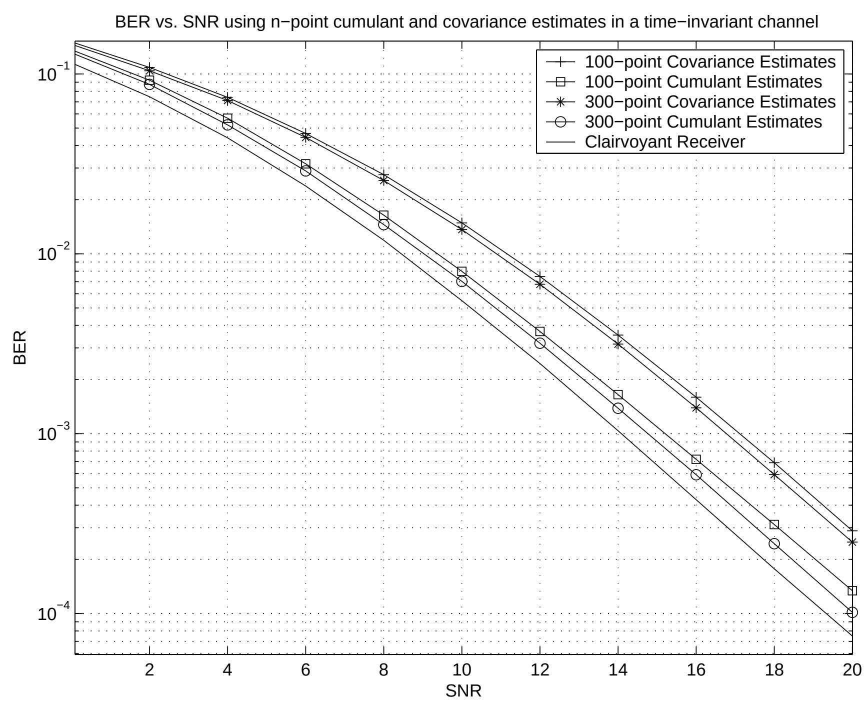

Fig. 4. Time Invariant Channel. BER vs. SNR using 100 and 300 point cumulant estimates.

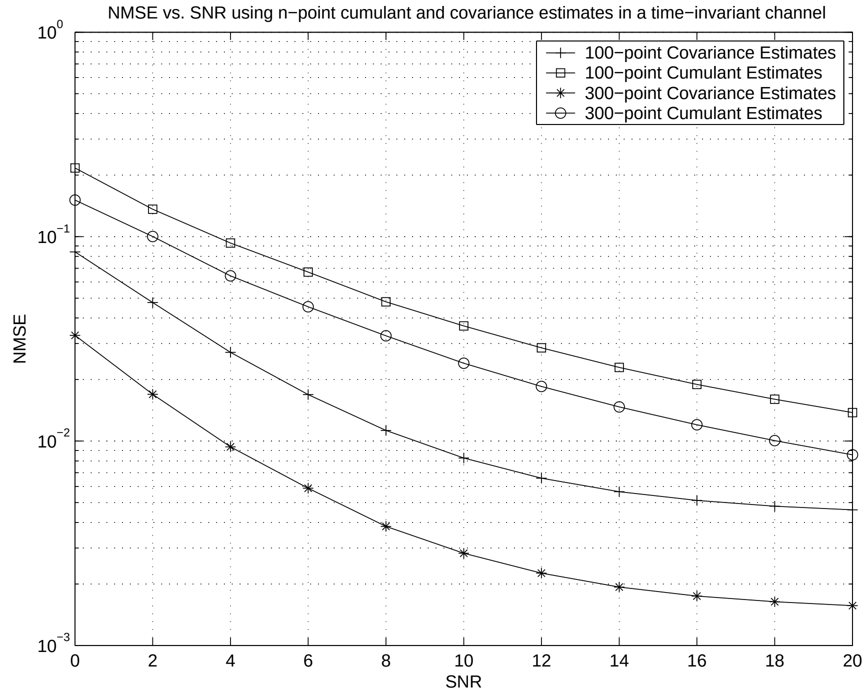

Fig. 5. Time Invariant Channel. BER vs. SNR using 100 and 300 point cumulant estimates.

of higher order cumulants, and the elimination of the requirement for data precoding. This improved performance is obtained, of course, at the price of increased complexity.

B. Time-Varying Channel

The efficacy of the proposed channel estimation algorithm is now examined when used in a time-varying fading environment. The example is based on two transmit antennas and the Alamouti STBC. The data is sampled at a rate of 20 MHz and is modulated using a carrier with a frequency of 5.5 GHz. The mobile is assumed to be moving at 100 km/h. Clearly, such channels are of practical importance and better reflect the real world than the block fading model used in Section V-A.

For each Monte Carlo run, 2×10^6 bits are corrupted by the time varying channel. The channel estimate for a particular window of data is obtained from the cumulant estimate resulting from that same window. Channel estimates in one window are fixed, but change from window to window. To correctly identify the channels, a pilot symbol is inserted at the beginning of every window. We focus here on the BER, which is more meaningful in terms of practical implications. The results are averaged over 250 Monte Carlo runs.

The BER for window sizes of 50, 100, 300 and 500 are shown in Fig. 6. Unlike in the block fading example, the estimates obtained using 300-point windows outperform those obtained using 500-point

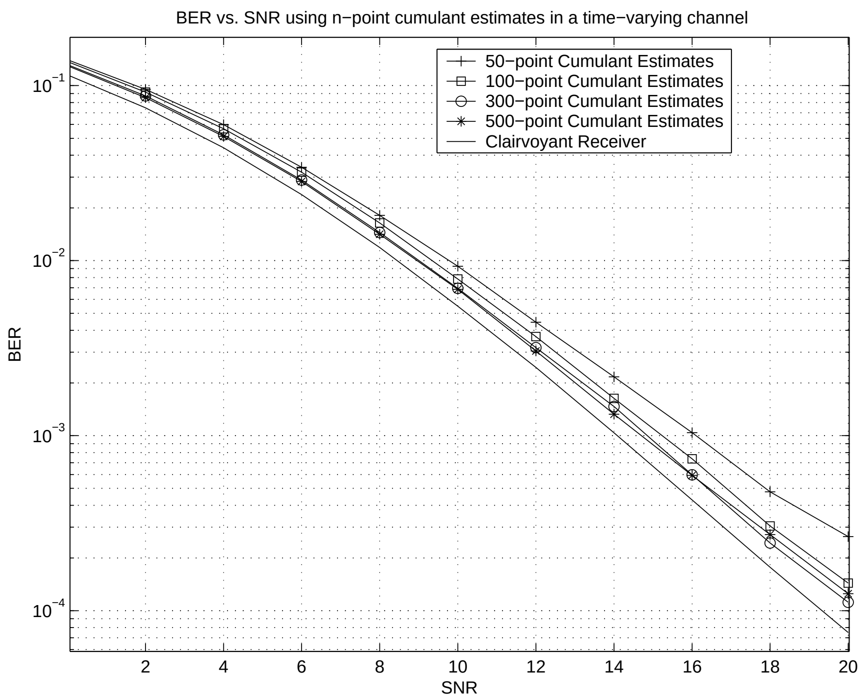

Fig. 6. Time-Varying Channel. BER vs. SNR using 50, 100, 300 and 500 point cumulant estimates.

windows at higher SNR. This occurs because when the channel changes over time, the cumulant estimate does not necessarily become more accurate for longer inputs. The variation in signal statistics therefore imposes a limit on the maximum number of points that can be used in cumulant estimates. The faster these channel variations, the fewer the points that can be used.

In Figure 7, we compare BER performance of our scheme to that that of [14] in time-varying channels. This comparison is important, since covariance estimates are assumed to require less data than cumulant estimates; it is thus conceivable that the comparison could change in time-varying channels. The figure demonstrates, however, that the number of points required to estimate cumulants and then to obtain accurate channel estimates, are sufficiently low to not be affected by the varying channel. The BER superiority of our proposed scheme over the scheme in [14] holds in this scenario.

VI. Conclusions

This paper presents a blind channel estimation algorithm for orthogonal space-time block coded data in the important MISO situation, i.e., in systems using only a single receive antenna. It is shown that the algorithm outperforms in terms of BER the only other existing algorithm applicable to this scenario. A significant cost of the algorithm is the complexity involved in estimating the required cumulants. Cumulant estimates are generally assumed to be impractical since they require too many samples to be effective. Very

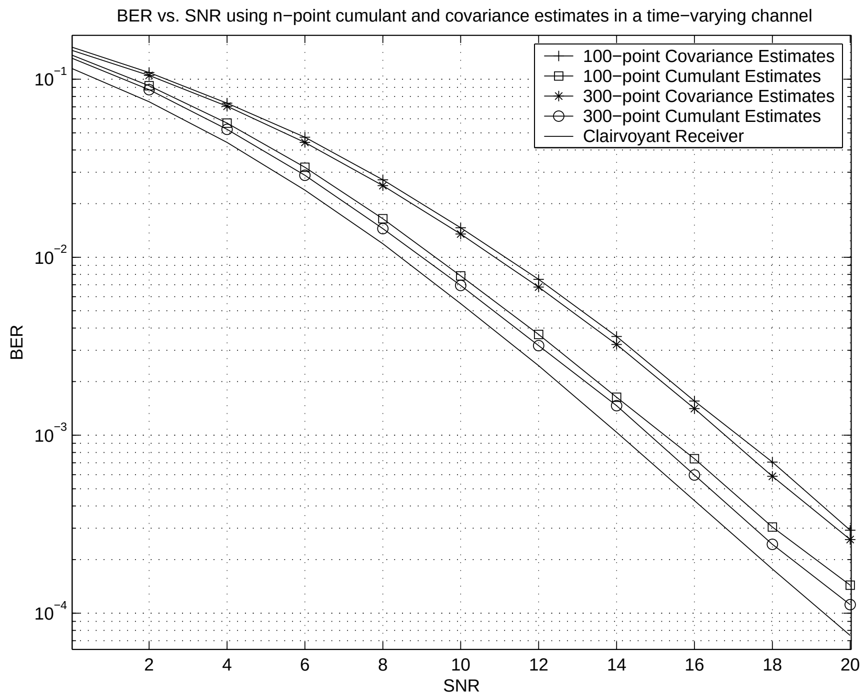

Fig. 7. Time-Varying Channel. BER vs. SNR using 50, 100, 300 and 500 point cumulant estimates.

good estimation results, however, are obtained when using as few as $100 - 300$ points in the cumulant estimates. The numerical simulations prove that even if using the the sample number of samples, the cumulant-based scheme presented here outperforms the previously proposed scheme based on covariance estimates.

APPENDIX I
CHANNEL MAGNITUDES

The channel magnitudes can be obtained from the non-zero eigenvalues λ_j as follows:

$$\begin{aligned} \lambda_j &= \gamma_4 ||\mathbf{h}||_F^2 f_j, \\ &= \gamma_4 |h_{kj}|^2 ||\mathbf{h}||_F^2, \end{aligned} \tag{31}$$

where f_j represents the the j^{th} non-zero diagonal entry of matrix $\mathbf{F_k}$ defined in (20) and (21) for complex and real data, respectively. $||\mathbf{h}||_F^2$ is found by summing the non-zero eigenvalues $\lambda_\mathbf{j}$ of each cumulant matrix, $\mathbf{C}_4^{[k]}$:

$$\sum_{k=1}^{2K}\sum_{j=1}^{L}\lambda_j = \sum_{k=1}^{2K}\sum_{j=1}^{L}(\gamma_4|h_{kj}|^2||\mathbf{h}||_F^2) = \alpha\gamma_4||\mathbf{h}||_F^4. \tag{32}$$

$$\Rightarrow ||\mathbf{h}||_F^2 = \frac{1}{\sqrt{\frac{1}{\alpha}\gamma_4\sum_{j=1}^{L}\lambda_j}}. \tag{33}$$

Here, α is a parameter dependent on the constellation: $\alpha = 2$ for real and $\alpha = 4$ for complex constellations.

APPENDIX II
NMSE ANALYSIS

In this appendix we show that for the Alamouti STBC with one receive antenna and for a particular channel pair, the NMSE can be approximated by (30). We assume here a high SNR region, and thus channel estimation errors are due only to estimation errors due to a finite number of samples, and are insensitive to noise. In this section, we do not separate the real and imaginary parts of the received signal, and use the model

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} h_1 s_1 + h_2 s_2 \\ -h_1 s_2^* + h_2 s_1^* \end{bmatrix} + \mathbf{w}, \tag{34}$$

$$\Rightarrow \mathbf{r}' = \begin{bmatrix} r_1 \\ r_2^* \end{bmatrix} = \mathbf{H}_c \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \mathbf{w}, \tag{35}$$

where

$$\mathbf{H}_c = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix}. \tag{36}$$

Without loss of generality, we define $\mathbf{v_1}$ and $\mathbf{v_2}$ as the normalized eigenvectors of the cumulant matrix $\mathbf{C}_4$, such that $\mathbf{v}_1^H\mathbf{v}_1 = \mathbf{v}_2^H\mathbf{v}_2 = 1$. $\mathbf{v}_1$ and $\mathbf{v}_2$ correspond to the scaled columns of the channel matrix $\mathbf{H}_c$, such that

$$\mathbf{v}_1 = \frac{1}{\sqrt{|h_0|^2 + |h_1|^2}} \begin{bmatrix} h_0 \\ h_1^* \end{bmatrix}, \qquad \mathbf{v}_1 = \frac{1}{\sqrt{|h_0|^2 + |h_1|^2}} \begin{bmatrix} h_1 \\ -h_0^* \end{bmatrix}. \tag{37}$$

Defining $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ as the eigenvectors of the estimated cumulant matrix, $\hat{\mathbf{C}}_4$, and $\delta_{v_j} = \hat{\mathbf{v}}_j - \mathbf{v}_j$ as the error between them, the NMSE of the estimated channels is exactly $\delta_{v_1}^H \delta_{v_1} = \delta_{v_2}^H \delta_{v_2}$. Because the two are equivalent, we will focus on the error in the first eigenvector, $\delta_{v_1}^H \delta_{v_1}$.

In [19], Yuen and Friedlander derive the covariance matrix of the error between the estimated and true eigenvectors of a cumulant matrix. The expression is valid in cases where the cumulant matrix has distinct values. Applied to the problem in this work, the expression becomes

$$\delta_{v_1}^H \delta_{v_1} = \frac{1}{(\alpha_1 - \alpha_2)^2} \sum_{a_1=1}^{2} \sum_{a_2=1}^{2} \sum_{b_1=1}^{2} \sum_{b_2=1}^{2} v_{2,a_1}^* v_{1,a_2} v_{2,b_1} v_{1,b_2}^* E\{(\mathbf{C}_4 - \hat{\mathbf{C}}_4)_{a_1 a_2} (\mathbf{C}_4 - \hat{\mathbf{C}}_4)_{b_1 b_2}, \tag{38}$$

where the quantity $v_{i,j}$ is defined as the j^{th} element of vector i, and $(\mathbf{C}_4 - \hat{\mathbf{C}}_4)_{ij}$ is the ij^{th} element of $(\mathbf{C}_4 - \hat{\mathbf{C}}_4)$. $\mathbf{C}_4$ is as defined in (17).

In [20], Porat and Friedlander derive expressions for the covariance matrix of fourth-order sample cumulants of zero-mean and symmetrically distributed signals. The analysis in [20] assumes that a large number of data points, B, are used in the estimation of the cumulant matrix. In this analysis, the input to the forth-order cumulants is the received signal in (II). This signal $\mathbf{r}$ is both zero-mean and symmetric for both BPSK and QAM input signals $\mathbf{s}$, and thus the expression derived in [20] is valid. It is repeated here for convenience.

$$\begin{aligned}
&B \times E\left\{[c_4(k_1,k_2,l_1^*,l_2^*) - \hat{c}_4(k_1,k_2,l_1^*,l_2^*)] \cdot [c_4(m_1,m_2,n_1^*,n_2^*) - \hat{c}_4(m_1,m_2,n_1^*,n_2^*)]\right\} \\
&\approx \mu_8(k_1,k_2,m_1,m_2,l_1^*,l_2^*,n_2^*,n_2^*) \\
&\quad - \textstyle\sum_{p=1}^{2} \sum_{q=1}^{2} \mu_2(m_{3-p}, n_{3-q}^*)\, \mu_6(k_1,k_2,m_p,l_1^*,l_2^*,n_q^*) \\
&\quad - \textstyle\sum_{i=1}^{2} \sum_{j=1}^{2} \mu_2(k_{3-i}^*, l_{3-j}^*)\, \mu_6(m_1,m_2,k_i,n_1^*,n_2^*,l_j^*) \\
&\quad + \textstyle\sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{p=1}^{2} \sum_{q=1}^{2} \mu_2(m_{3-p}, n_{3-q}^*)\, \mu_2(k_{3-i}, l_{3-j}^*)\, \mu_4(k_i, m_p, l_j^*, n_q^*) \\
&\quad - [[u_4(k_1,k_2,l_1^*,l_2^*) - 2u_2(k_1,l_1^*)\, u_2(k_2,l_2^*) - 2u_2(k_1,l_2^*)\, u_2(k_2,l_1^*)] \\
&\qquad \cdot [u_4(m_1,m_2,n_1^*,n_2^*) - 2u_2(m_1,n_1^*)\, u_2(m_2,n_2^*) - 2u_2(m_1,n_2^*)\, u_2(m_2,n_1^*)]]\,.
\end{aligned} \tag{39}$$

To determine the expression in (38), the 16 corresponding cumulant covariances must be determined using (39). This can be simplified using the fact that $(\mathbf{C}_4 - \hat{\mathbf{C}}_4)$ is Hermitian, and thus, for example

$$(\mathbf{C}_4 - \hat{\mathbf{C}}_4)_{11} (\mathbf{C}_4 - \hat{\mathbf{C}}_4)_{12}^* = (\mathbf{C}_4 - \hat{\mathbf{C}}_4)_{21} (\mathbf{C}_4 - \hat{\mathbf{C}}_4)_{11}^*. \tag{40}$$

The calculations for the necessary cumulant covariances are tedious but straightforward, and the results are given below:

$$E\{(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{11}(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{11}\} = \gamma_4^2(h_0^4h_1^{*4}+h_0^{*4}h_1^4+18|h_0|^4|h_1|^4+8|h_0|^2|h_1|^6+8|h_0|^6|h_1|^2) \tag{41}$$

$$E\{(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{22}(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{22}\} = \gamma_4^2(h_0^4h_1^{*4}+h_0^{*4}h_1^4+2|h_0|^4|h_1|^4) \tag{42}$$

$$\begin{aligned}E\{(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{11}(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{12}\} = &\gamma_4^2(-h_0^5h_1^{*3}+h_0^{*3}h_1^5-3h_0h_1|h_0|^4|h_1|^2 \\ &+3h_0h_1|h_0|^2|h_1|^4+2h_0h_1|h_1|^6-2h_0h_1|h_1|^6)\end{aligned} \tag{43}$$

$$E\{(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{12}(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{12}^*\} = \gamma_4^2(-h_0^4h_1^{*4}-h_0^{*4}h_1^4+3|h_0|^6|h_1|^2+3|h_0|^2|h_1|^6+2|h_0|^4|h_1|^4) \tag{44}$$

$$E\{(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{11}(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{22}\} = \gamma_4^2(-h_0^4h_1^{*4}-h_0^{*4}h_1^4-2|h_0|^4|h_1|^4) \tag{45}$$

$$E\{(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{12}(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{12}\} = \gamma_4^2(-6h_0^2h_1^2|h_0|^2|h_1|^2-2h_0^2h_1^2(|h_0|^4+|h_1|^4)+h_0^6h_1^{*2}+h_0^{*2}h_1^6) \tag{46}$$

$$E\{(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{22}(\mathbf{C}_4-\hat{\mathbf{C}}_4)_{12}\} = \gamma_4^2(h_0^5h_1^{*3}-h_0^{*3}h_1^5-h_0h_1|h_0|^2|h_1|^4+h_0h_1|h_0|^4|h_1|^2) \tag{47}$$

Using (41) - (47), the expression for the eigenvalues of the cumulant matrix given in (32) and using (37) are substituted in (38), results in

$$\text{NMSE} \approx \frac{|h_0|^4+3|h_0|^2|h_1|^2+|h_1|^4}{B(|h_0|^2-|h_1|)^2)^2} \tag{48}$$

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